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Discrete gambles: Theoretical study of optimal bet allocations for the expo-power utility gambler

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Abstract

Given a gamble opportunity with a positive expected return, the question to be asked is: How much money should I gamble?

In this thesis, we study how a gambler who holds a general expo-power utility function allocates wealth in discrete gambles. The analyzed gambles are the following: a single binary gamble, equivalent to betting on a tennis match, a double binary gamble, equivalent to betting on two simultaneous tennis matches, and a discrete financial market gamble, equivalent to betting on a financial asset with a discrete outcome distribution. We derive analytical expressions that can be used to approximate utility-optimal bet allocations for any specified expo-power utility function. We propose a general approximation formula to obtain bet allocations at large wealth levels. The Newton-Raphson method is applied to locate optimal bet allocations at lower wealth levels.

The main motivation for the thesis is to provide analytical results that enable von Neumann-Morgenstern rational gamblers to establish optimal bet allocation strategies.

Keywords: optimal bet allocation, discrete gambles, expo power utility function

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1. Introduction

Perhaps the most famous bet allocations problem, so far in history, is the St. Petersburg paradox[1] presented by Nicolas Bernoulli in the early 18th century. The St. Petersburg game is setup so that the expected arithmetic value of playing the game is infinite. However, the probability of winning an infinite sum is infinitely small. How much shall one pay to play the St. Petersburg game? Bernoulli (1738) was first to publish the idea that wealth exhibits diminishing marginal utility. In Bernoulli's paper *Exposition of a New Theory on the Measurement of Risk* the log utility function is introduced as a solution to the St. Petersburg game. By taking log over the game's expected value the utility of the game is finite. Consequently, the log-utility gambler will only pay a finite sum to play the game. Later studies have shown that every unbounded utility function has its own St. Petersburg paradox emerging by adjusting the payoff structure of the game (Menger 1934). For that reason, utility functions are normally assumed to have an upper bound for wealth. The utility theory was further developed by von-Neumann Morgenstern in the frequently quoted book *Theory of Games and Economic Behavior*. The researchers postulated axioms of preference relations over discrete lotteries. They showed that if their axioms hold, there exists a unique real preference function *U* deciding the gambler's preferences in the selection of lotteries. Kelly (1956), a decade later, introduced the growth optimal portfolio (GOP) criteria as an alternative method to solve bet allocation problems. The GOP bet allocation maximizes the expected geometric portfolio growth, equivalent to maximizing long-run portfolio growth. The GOP research is a niche area in economics and commonly associated with bet allocation strategies.

The idea of this paper is to merge parts of the GOP research with the expected utility framework. We will apply von-Neumann Morgenstern (VNM) utility optimization techniques on a set of discrete gambles which have previously been studied in the GOP research. The motivation is to analyze optimal bet allocations for gamblers whose preferences are VNM-rational. GOP preferences imply that gamblers are always willing to accept higher (geometric) mean return for greater variance. With the widespread usage of applying modern portfolio theory to investment decisions, we think it is clear that individuals are willing to reduce investment returns in order to reduce the variance risk.

[1] The game is a series of coin tosses, and the gambler doubles his winning for every next toss until a pre-selected side of the coin is realized. Thus, the expected value of the game is $\sum_{i=1}^{\infty} \frac{1}{2^i} 2^i = \infty$.

We choose the general expo-power (EP) utility function to describe gambling preferences. The EP utility function is able to describe increasing and decreasing risk aversion preferences. The risk aversion preferences are set by the parameters of the utility function. The main advantage of the EP utility function is that only two parameters are needed to specify the function. We will apply the EP utility function on a set of discrete gambles and then derive general analytical expressions enabling us to approximate utility optimal bet allocations. Furthermore, we propose a general approximation formula valid for large wealth values and show how the Newton Raphson method is used to locate utility-optimal bet allocations.

1.1. Problem statement

Before the gamble, the gambler needs to decide how much wealth to allocate to the gamble. The bet allocation problem is subjective to the gambler in question if the choice of bet allocation depends on private risk preferences. An argument against VNM-rationality is that gambling preferences could be described by a universally accepted criterion. Such as the growth optimal portfolio criterion, that the bet allocation is set in order to maximize geometric mean return. We find the universal description of gambling preferences to be faulty. The main critique against the growth optimal criterion is that the gambler has a constant relative risk aversion with such a strategy. The GOP gambler will always allocate the same relative fraction of wealth to equivalent gambles, unaffected by the current wealth level. Given a binary gamble with a 55% winning percentage the growth optimal gambler allocates x% of current wealth to the bet irrespectively if the current wealth level is $1 or $1 billion.

We assume that the gambler is willing to reduce investment return in order to reduce variance.

VNM-utility functions are categorized in utility families according to functional form. We restrict our study to the expo-power (EP) utility function mainly due to its simple functional form yet flexible preference set. The EP utility function has only two parameters simplifying estimation procedures in future empirical works, and its exponential form is simplifying optimization results.

Given the set of discrete gambles, the specific purpose of this thesis is to answer the question:

How to find expected utility optimal bet allocations to a set of discrete gambles for a gambler with an expo-power utility function?

The general EP utility function will be applied to derive analytical expressions that can be utilized to approximate utility optimal bet allocations for the given set of discrete gambles. The discrete gambles are chosen to reflect popular money gambles. Since the EP utility function is general in form, the analytical results are likewise general. We will demonstrate how the Newton-Raphson method can be used to approximate optimal bet allocations for a specified EP utility function. We also put forward an approximation formula which lessens the computer burden of locating optimal bet allocations at large wealth levels. The approximation techniques are necessary because there are no general explicit solutions to solve for optimal bet allocations in the given gambles.

1.2. Motivation

This thesis should be viewed as an extension of previous research about bet allocation strategies in discrete gambles. We focus our thesis on a set of bet allocation problem commonly associated with the GOP research but put the allocation problem in a VNM-utility framework, and more specifically the gambles are played in the general EP utility function. This is a neglected research area probably due to the small size of traded markets with discrete outcome distributions, mainly the gambling markets. We think the thesis-subject is growing in importance especially given the growing presence of derivative products with discrete outcome distribution such as binary options.

A further motivation for conducting this study is the strong growth of automated betting algorithms applied in online gambling markets. Automated betting algorithms place high frequency bets on deterministic and statistical arbitrages. As their counterparts in financial markets, the presence of betting algorithms increase the liquidity in gambling markets and put pressure on betting exchanges to lower transaction costs. To utilize the results in this paper, the gambler first need to specify an EP utility function. The effort of specifying and optimizing such a utility function may only be economically meaningful if the gambler engages in large scale gambling operations. Going forward in the paper, we will provide short notes on how utility functions could be estimated in order to serve those readers who are interested to apply the general results on a practical level. We always think it is useful to discuss how theoretical results can add practical value.

2. Background

This section is mainly a summary of general economic theory related to assumptions made in the results. Here, we further motivate and clarify our assumptions behind the results, such as the VNM-utility assumption. If alternative assumptions were possible, or if chosen assumptions are arguable, we will motivate those choices. Subsequent paragraphs include parts of von-Neumann Morgenstern expected utility theorem, Arrow-Pratt measures of risk aversion, and expected utility theory. Experienced readers are advised to shortcut the section.

We start with an exposition of our utility assumption. Von Neumann-Morgenstern (VNM) proved that if their preference axioms hold, there exist a real preference function U on outcome x so that,

$$\mathrm{x} > y \ \ if \ E[U(\mathrm{x})] > E[U(\mathrm{y})]$$

$$\mathrm{E[u(x)] = p_1u(x_1) + p_2u(x_2) + p_3u(x_3) + \cdots + p_nu(x_n)},$$

with outcomes $\mathrm{x_1, x_2, \dots, x_n}$ and their respective probabilities $\mathrm{p_1, p_2, \dots, p_n}$. Most people recognize VNM's first axiom of *completeness* that says that agents have a preference ranking on a set of outcomes. The second axiom is of transitivity, if X is preferred to Y and Y to Z, then X is preferred to Z. The most debated VNM-axiom is the one referred to as the *independence axiom*. If X is preferred to Y, then for Z in the probability interval [0 1], we have $\mathrm{pX + (1-p)Z}$ preferred to $\mathrm{pY + (1-p)Z}$. This preference axiom contradicts the empirical result in the famous behavioral experiment *Allais choice problem*[2] (1953) designed by Maurice Allais. The experiment showed that most individuals hold preferences that are inconsistent with the independence axiom given the experiment setup. Another VNM-critical research paper that has received attention is Rabin (2000) who demonstrated that a utility function with modest risk aversion for lower stakes could lead to absurdly high risk aversion

[2] The Allais paradox consists of two sessions. In the first session, the subjects are choosing between option A: winning $1 million with certainty, or option B: $5 million with probability 10% (P10%) , $1 million with P89% and nothing with P1%.. In session two the rules are changed, option A is: Nothing with P89%, $1 million with P11%, and option B is: Nothing with P90%, $5 million with 10%. VNM-consistent behavior is to choose either AA or BB. But a significant fraction of the subjects preferred the combination AB. As a footnote to the footnote, the author of this paper holds preferences for combination BB.

for larger stakes. But as pointed out by Cox and Sadiraj (2001), the utility function in Rabin's paper defines the lottery outcome as the final wealth variable, ignoring previously accumulated wealth. We agree with Rubensten in his comments[3] to Rabin's paper: *"The idea that a person who does not have an apple today is ready to surrender two apples tomorrow in order to get one today is not implausible. (p3)"* However, axioms cannot be proven true and neither can they be proven false. At the risk of sounding naive, we think that the main criterion for using the expected utility axioms in decision analysis should be plausibility. The expected utility theorem is a plausible methodology if preferences can be assumed to be VMN-rational. Is it plausible to assume that gambling preferences are VMM-rational in discrete gambles?

The general answer must be no. Behavioral experiments have clearly shown that the independence axiom does not hold empirically in lab settings. However, in a traded gambling market an opportunity equivalent to the Allais choice problem is non-existent. In similar fashion as Samuelsson (1960) argues that the St Petersburg paradox can be solved by concluding that the game will never be offered. Since the expected profit of offering the St Petersburg lottery is an infinite loss for the bookmaker. We think it is reasonable to assume that gamble opportunities similar to the ones offered in the Allais choice problem never occur in gambling markets. We acknowledge the existence of arbitrage opportunities in gambling markets. However, the probability of successfully implement arbitrage strategies is less than 100% certain. The pure arbitrage displayed in Allais choice problem does not reflect true market conditions. Conclusively, we feel certain to assume that gambling preferences in gambling markets are VNM-rational.

The next paragraph takes on Arrow-Pratt measures of risk aversion and their applications to this paper. The aim is to prepare the reader for a discussion of logical properties of utility functions directly related to the Arrow-Pratt measures which will be further elaborated in the results section. Moreover, we aim to clarify our arguments for the choice of the expo-power utility function. The text also serves as a practical note on how risk preferences can be measured.

Utility is not a measurable physical unit and thus it arises some difficulties moving the utility concept to a practical level. One may solve the difficulties by arguing that economic agents know their own utility functions. Or one may argue that utility functions can be estimated.

[3] Comments on the Risk and Time Preferences in Economics. (2001)

Or accept that both arguments are true. One method to estimate a utility function was introduced by Arrow-Pratt (1964 and 1965) with their measures of risk aversion. Their risk aversion measure is simply the second derivative of the utility function divided by its first derivative. Due to its mathematical construction, the Arrow Pratt risk aversion measures are invariant of positive affine transformation of the utility function. The measures also serve as a link between the certain equivalent and the utility function. The VNM-utility function is invariant to positive affine transformations so it was important that any consistent risk aversion measure would have that same property. The property of being invariant to positive affine transformations implies that we can add and/or multiply a positive constant to the utility function without changing the curvature of the function. The certain equivalent is the risk free payout that yields the same utility to the individual as his/her utility of participating in a gamble. So to illustrate the certainty equivalent algebraically:

$$U(w_0 + E(x) - c) = E[U(w_0 + x)]$$

The individual is indifferent between receiving the certainty equivalent $w_0 + E(x) - c$, and accepting the risky gamble $w_0 + x$, where the random variable x has variance σ^2 and expected value μ. Thus, the risk premium is $E(x) - c$, i.e. the wealth the individual sacrifice in order to obtain a risk free endowment. Arrow-Pratt approximated both sides of the expression with a Taylor series expansion around $w = w_0 + \mu$,

$$U(w) - cU'(w) + R_2 = U(w) + E(x-\mu)U'(w) + \frac{1}{2!}E(x-\mu)^2U''(w) + r_3$$

, where R_2 is a remainder term including second order terms and R_3 is a third-order remainder term. Arrow and Pratt assumed that the remainders could be ignored, and after some algebraic manipulation they received the following approximation:

$$c = -0.5\sigma^2 \frac{u''(w)}{u'(w)}$$

$$A(w) = -\frac{u''(w)}{u'(w)}$$

Where $A(x)$ is the Arrow-Pratt absolute risk aversion measure (ARA), and $A(x) * w$ is defined as the relative risk aversion measure (RRA). Notice that in both measures wealth is the expected value of wealth. Decreasing absolute risk aversion (DARA) if wealth increases results in a decreasing risk premium and hence the gambler is more willing to allocate a larger absolute sum to the lottery. Decreasing relative risk aversion (DRRA) is almost equivalent to DARA with the only difference that the decision maker is allocating a larger relative fraction of current wealth to the lottery. Empirical support for DRRA preferences are found in Ogaki, Shang (2004), Graves (1979). However, in theoretical models there is a frequent usage of constant relative risk aversion (CRRA) preferences. This is mainly a model simplification. Intuitively most people agree on that the individual who owns no apple to eat today puts a greater weight on the downside risk than the individual who owns the apple shop. Nevertheless, the CRRA functions serve as fairly good approximations in many wealth ranges. Empirical works of estimating ARA risk aversion parameters are quite contradicting. Meyer (2006) offers an explanation by pointing out that researchers have applied different data set, different utility assumptions, and different estimation procedures. Meyer concludes that the main reason for the discrepant results is due to the differences in the definition of the argument variable. We think that Meyer's argument is intuitively clear. If the frequently used input variable *wealth* lacks a shared definition, the researchers will have to subjectively define the variable increasing the chances that risk aversion estimates will differ from each other.

Given an absolute risk aversion function one can obtain the functional form of the utility function trough differential equation techniques, the ansatz would be:

$$U(w) = \int e^{-\int \frac{u''(w)}{u'(w)}}$$

The parameters in the risk aversion function follow trough to the utility function. If one wishes to estimate a utility function, the parameters can be estimated by the risk aversion function. Notice that the Arrow Pratt risk aversion measure is an approximation thus only valid for small gambles in a region around w. The approximation is only exact if the gamble has a symmetric probability distribution, equivalent to a fair gamble. A fair gamble is one in which the bet size equals the expected return of the gamble. In the result section, we approximate optimal bet allocations to a given parameter set. While no assumption is made, the parameter choices indicate that the gamble is non-symmetric. In the case of a non-symmetric gamble it will bias the estimations of parameters in the ARA-measures. Parameters then need to be estimated directly from utility elicitation methods. However, the gambler may be advised to apply a prudent view estimating risk aversion parameters and assume that a gamble is fair in order to avoid overconfidence bias. That finishes the Arrow-Pratt section; we think it is now clear how our general results soon presented can be utilized in practical work.

Below we elaborate our arguments for the choice of the expo-power (EP) utility function. The EP utility function will later be applied to derive analytical expressions that enable us to obtain optimal bet allocations for a given set of gambles. The EP utility family was introduced by Saha (1993), and proposed as an improvement of the hyperbolic absolute risk aversion (HARA) utility family (Merton 1971). The EP utility function is defined as,

$$u(w) = \theta - exp(-\beta * w^{\alpha})$$

, in the parameter range $\alpha\beta > 0$ and $\theta > 1$. The θ parameter is an optional constant, and is not part of the function's absolute risk aversion measure. The EP utility function's absolute risk aversion measure is defined in a two-parameter form, where

$$A(w) = \frac{(1-\alpha+\alpha*\beta*w^{\alpha})}{w}.$$

Likewise as the HARA utility function, the EP utility function is able to fit increasing and decreasing absolute and relative risk aversion preferences. So to complete the comparison, the HARA utility function is defined as, $u(x) = \frac{1-\gamma}{\gamma}(\frac{\beta w}{1-\gamma} + n)^{\gamma}$ and its absolute risk aversion function is $A(w) = \frac{1}{\frac{w}{1-\gamma}+\frac{n}{\beta}}$. From a practical perspective the EP utility function is simpler to apply than the HARA utility function. We prefer working with a utility function that includes less parameters but still is able to fit a full range of risk preferences. For this reason, we choose the EP utility function over the HARA utility function.

2.1. Relevant work

Here we focus on explaining the research that has influenced us and which set the direction for the thesis. We will go trough important works in the growth optimal portfolio (GOP) research area and further motivate our decision to adopt a VNM-utility framework to discrete gambling decisions.

The discrete gambles presented in the result section are inspired by traded gambling markets. For example, the first gamble is a single binary gamble equivalent to betting on the winner in a tennis match. In the VNM-utility framework, the player faces the problem to decide the bet allocation that maximizes his/her expected utility with respect to future outcomes. The similar tennis gamble has been analyzed in GOP research. The binary gamble problem is then formulized to find the bet allocation that maximizes the portfolio's geometric mean return which is equivalent to maximizing long-run portfolio growth. Kelly (1956) is usually associated as the founder of the GOP research area. Although Kelly was solving an information theoretical problem of channel capacity, he did it by finding the bet allocation that maximizes the exponential growth rate of the gambler's wealth. In Kelly's problem, the gambler is facing a bet with even odds, hence a 100% return on every successful bet outcome. Thus, the gambler's wealth level after N bets is:

$$w_N = (1 + l)^V (1 - l)^L w_0$$

, where $w_N$is the gambler's wealth after N bets, and w_0 is his/her initial wealth. The bet allocation, the fraction of wealth placed on the bet, is denoted l. The number of wins and losses after N bets are denoted V respectively L. The exponential growth rate of gambler's wealth is expressed as:

$$G = \lim_{n \to \infty} \frac{1}{N} \left[\frac{V}{N} \log(1 + l) + \frac{L}{N} \log(1 - l) \right] = plog(1 + l) + qlog(1 - l)$$

The probability of winning, respectively losing, the bet is denoted p and q. The logarithm is of base two. Kelly found that the maximum exponential growth rate of the gambler's wealth is obtained by allocating a fraction of wealth to the bet as $\mathrm{l} = p - q$. The maximum growth rate is then:

$$G_{max} = (p + q) \log(1 - q + 1 - p) + plog(p) + qlog(q) = 1 + plog(p) + qlog(q).$$

In the results section, we solve a similar type of gamble. However, we apply the general expo-power utility function to the gamble problem to obtain a general solution. Unfortunately, the mathematic properties of the "EP-utility gamble" do not allow us to solve explicitly for optimal bet allocation as is done in the GOP example.

Going back to the GOP research, Thorpe (1969) with contribution of J. Holladay, generalized the binary gamble to a series of independent gambles. Given the same payoff structure as in previous example, the gambler's wealth level after N is then,

$$W_N = \prod_{i=0}^{N} (1 + \frac{l_i}{W_{i-1}} X_i)$$

, where X_i is a random variable which determines the outcome of gamble i. Notice that the binary gambles still display even odds. However, the win-loss probabilities are captured by the random variable. The exponential growth rate of the gambler's capital is expressed as,

$$G = \sum_{i=0}^{N} \log\,(1 + l_i X_i)$$

The optimal bet allocation l_i is a function of previous $i - 1$ gambles. Thus the variables l_i and X_i are independent. Thorpe proved that there exist a G_{max} in the N-gamble. For each i gamble there is an optimal bet allocation l_i^* so that $\mathrm{E}[\sum_{i=0}^{N} \log\,(1 + l_i X_i)$ reaches a unique maximum. We generalize a quite similar N-gamble in the result section, with the difference that we look at a financial market bet, and again, we apply the gamble to the EP-utility function. Notice that the presentation of the results of Kelly (1956) and Thorpe (1969) has a slightly different appearance in this paper in contrast to their original works. The purpose was to a setup coherent structure in the thesis, and not mix notation. Before moving on to new subjects, it is worthy to mention other pioneering papers in the GOP research area, for example: Breiman (1961), Hakansson (1971), and Markowitz (1976).

The observed reader may have noticed that the GOP strategy is equivalent to utility maximizing the VNM-log utility function. Therefore, the GOP research could be labeled as a subclass of the expected utility theorem. Below is short outline of further arguments for adopting a general utility approach for bet allocation problems in discrete gambles.

The GOP research is large and its advocators like to point out the rationale of growth optimizing the portfolio. However, the GOP strategy is not a universal criterion, as emphasized by Samuelsson (1963). Samuelsson showed that if the concave VNM-utility function rejects the single bet it also rejects the long sequence of the same bet[4].Utility theory has progressed since then to now include utility concepts so that rejecting the single bet does not necessarily mean rejecting the long sequence of bets. However, the method we studied of cutting up the variance of the gamble in n-independent risk portions to be able to accept the long sequence of bets, as in Rossi (1999), is hard to conceptualize in traded gambling markets. The additional risk of accepting a long sequence of gambles in incomplete markets is usually held by the gambler. In traded gambling markers, there exist yet no insurance opportunities to

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[4] The story goes that Samuelsson offered a colleague the 50% chance of winning \$200 or losing \$100. The colleague replied that he would accept a sequence of 100 such gambles. This challenged Samuelsson to write the paper about the fallacy of large number.

offload additional variance risk. We are certain that analyzing discrete gambles with traditional VNM-utility maximization techniques is a valid method.

3. Results

This section provides analytical results of our study of optimal bet allocations. We also demonstrate how to obtain approximation results of optimal bet allocation by using the Newton Raphson method. We use the expo power (EP) utility function for maximizing expected utility over future outcomes. To further emphasize the link between the Arrow-Pratt risk aversion measures and the VNM-utility function, we like to point out that the EP utility function is the solution to the differential equation: $u''(w) + u'(w)\frac{(1-\alpha+\alpha\beta w^{\alpha})}{w} = 0$. While the expo-utility function is $\mathrm{U}(w) = \theta - exp(-\beta w^{\alpha})$, in the parameter range $\alpha\beta > 0$ and $\theta > 1$.

The first discrete gamble to be analyzed is the previously described single binary gamble. The player receives net odds b for one of the outcomes. Thus the player's net win is b multiplied by bet allocation l if the winning outcome is realized. The winning outcome is realized with probability p, and the probability for the losing outcome is q. Player initially holds wealth w_o. Thus, the binary gamble is setup as,

$$E(U[w_1(l)]) = p[(\theta - \exp(-\beta(w_o + bw_o l)^{\alpha}) + q[\theta - \exp(-\beta(w_o - w_o l)^{\alpha}) \quad (1)$$

, where wealth in next period, period 1 denoted w_1, is a function of the relative bet allocation l. By differentiating expected utility w.r.t to bet allocation l and after some algebraic manipulation, the following expression holds:

$$U'(l) = \alpha w_o \beta[pb(w_o + bw_o l)^{\alpha-1}\exp(-\beta(w_o + bw_o l)^{\alpha}) - q(w_o - w_o l)^{\alpha-1}\exp(-\beta(w_o - w_o l)^{\alpha})]$$

$$(2)$$

There exist only implicit solutions to expression (2) due to the mathematical properties of the first order expression. The EP utility function is a continuous function and the input argument is a linear function. Furthermore, Thorpe's proof of the existence of a unique maximum in the N-dimensional discrete gambles for the he log-utility function indicates that EP-utility functions should attain unique maximums for these types of discrete gambles. However, there exist no general proof that the EP utility function attain a general unique maximum for N-dimensional discrete gambles since the second derivative of expression (1) is positive for certain ranges of parameter values. Intuitively, we are able guess a range of where maximum utility is attained for discrete gambles. We know the explicit formula to calculate the optimal bet allocation for the growth optimal gambler in the single binary gamble: $l_g = p - q$. Let us assume that the EP utility function is parameterized to display more risk averse preferences than the growth optimal function. We then realize that the EP utility gambler achieves a utility optimal gamble in the bet allocation range $0 < l^* < l_g$. Given this range, we can apply graphical analysis and/or the Newton-Raphson method to attain a close approximation of any optimal bet allocation value. By calculating second derivatives values around approximated critical points, we can be highly certain whether an optimal bet allocation has been approximated.

The Newton-Raphson method is an iterating formula to approximate the root to f(x) = 0 by choosing the next approximation as $x_{n+1} = x_n - \frac{f(x_n)}{f^{'}(x_n)}$. In able to apply the method, we set expression (2) to "be" the Newton-Raphsons' $f(x_n)$, and then set expression (3) to be $f'(x_n)$. The second derivative of the expected utility function w.r.t to l, is:

$$U^{''}(l) = \alpha w_0^2 \beta (b^2 p \exp\left(-\beta x_p^a\right) \left[x_p^{a-2}(a-1) - x_{p,0}^{2a-2} a\beta\right] + q \exp\left(-\beta x_q^a\right)[x_{q,0}^{a-2}(a-1) - x_{q,0}^{2a-2} a\beta])$$

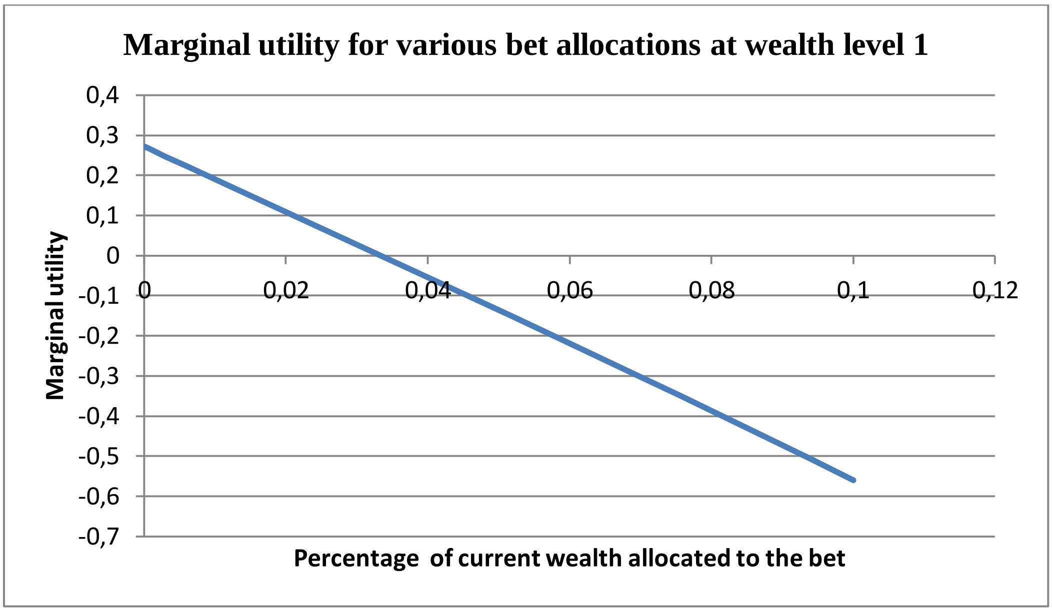
$$x_p = w_o + b w_o l \quad \text{and} \quad x_{q,0} = w_o - w_o l$$

We now demonstrate how an implicit solution to expression (2) is approximated with the Newton Raphson method. This example also shows the intuition behind our proposed approximation formula. We start by giving a parameter set to the utility function and the binary gamble:

p=0.55; q=0.45; b=1; $a = \beta = -1$

The parameters are chosen so that the gambler exhibits decreasing absolute and relative risk aversion as more wealth is accumulated. The wealth varies from 0.0001 to 1 000 000 0000 wealth units. Figure 1 displays a graph of the parameterized expression (2), showing how marginal utility is increasing and decreasing for various percentages of wealth allocated to the bet.

Figure 1

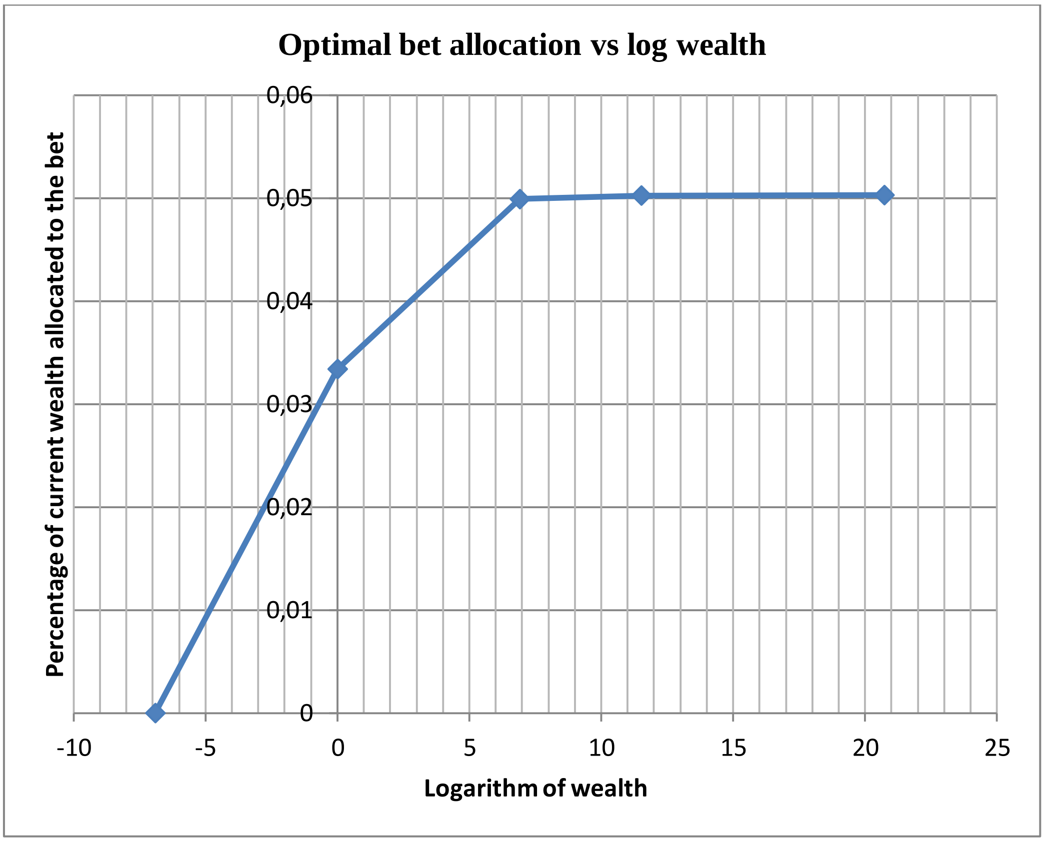


The y-axis represents the marginal utility of the parameterized utility function. The x-axis represents the percentage of wealth that is placed on the bet. We are locating an optimal bet allocation on a point on the x-axis of where marginal utility is approximately zero. The table below is the approximation results from using the Newton-Raphson method. The results are approximated to six digits. Approximation errors are displayed in appendix. Further below in figure 3, the approximation results are extrapolated against wealth.

Figure 2

Percentage of current wealth allocated to the single binary bet					
Wealth level, w_o	10^(-3)	1	10^3	10^6	10^9
Bet allocation, l	0.00000	0,03342	0.05010	0.05012	0.05012

Figure 3



In figure 3, the y-axis represents the relative bet allocation, and the x-axis represents current wealth level in base-10 logarithmic scale. As seen in the last figure, the optimal bet allocation is converging to a constant, close to 5%, as wealth goes to infinity. That pattern is explained by the Arrow-Pratt relative risk aversion measure (RRA). The RRA measure of the parameterized EP utility function is converging to 1-α, as wealth goes to infinity. The growth optimal log utility gambler would have bet a wealth fraction $p - q$, which is a 10% bet in this example. By using the growth optimal result as a proxy, we can be fairly certain about the interval of where to find a utility optimal bet allocation. As in this example, the bet allocation is found in the range 0 – 0.1 because the specified EP-utility gambler exhibits a higher degree of risk aversion than the growth optimal gambler. The growth optimal gambler has a constant RRA measure of one. We propose the following approximation formula to find the optimal bet allocation as wealth goes to infinity:

$$l^* = \frac{p - q}{1 - \alpha}$$

It works because the EP utility function exhibit a constant RRA measure as wealth goes to infinity. Thus, the EP utility optimal bet allocation is simply a constant fraction of the growth optimal bet allocation for large wealth levels.

We now extend the single binary gamble to situations where the gambler is betting at two independent gambles coinciding in time, a *double binary gamble*. The gambler's utility expression is:

$$U[x_1(l)] = p_1 p_2 U[w_o(1 + bl_1 + cl_2)) + q_1 q_2 U[w_o(1 - l_1 - l_2)] + p_1 q_2 U[w_o(1 + bl_1 - l_2)] +$$

$$q_1 p_2 U[w_o(1 - l_1 + cl_2)]$$

(3)

, where p,q for $i = 1,2$, are the probabilities for winning respectively losing the gamble, b and c are the winning odds, and U is the EP-utility function. Partial differentiation result of the utility function w.r.t to bet allocation l_1 is:

$$U_{l_1}(l_1, l_2) = \alpha\beta w_o[p_1 p_2 b\big(w_o(1 + bl_1 + cl_2)\big)^{\alpha-1} \exp(-\beta\big(w_o(1 + bl_1 + cl_2)\big)^{\alpha}) - q_1 q_2\big(w_o(1 - l_1 - l_2)\big)^{\alpha-1} \exp(-\beta\big(w_o(1 - l_1 - l_2)\big)^{\alpha}) + p_1 q_2 b\big(w_o(1 + bl_1 - l_2)\big)^{\alpha-1} \exp(-\beta\big(w_o(1 + bl_1 - l_2)\big)^{\alpha}) - q_1 p_2 \big(w_o(1 - l_1 + cl_2)\big)^{\alpha-1} \exp(-\beta\big(w_o(1 - l_1 + cl_2)\big)^{\alpha})].$$

(4)

The partial differentiation of expected utility w.r.t to l_2 is solved in analogous fashion. The multivariate Newton-Raphson method is useful to find the critical points l_1 and l_2. The multivariate method is analogous to Newton-Raphson single variable method. The analytical results of the Jacobian matrix are found in the appendix. The Newton-Raphson multivariate method is also further explained in appendix. Below, we approximate optimal bet allocations to the double binary gamble and display how bet allocations may vary for independent gambles. We use the same parameter values as in previous example:

$$p_1 = p_2 = 0.55; \quad q_1 = q_2 = 0.45; \quad \alpha = \beta = -1; \quad b = c = 1$$

The two coinciding gambles are symmetric in odds and probabilities so optimal bet allocations, l_1 and l_2, are identical.

Figure 4

Percentage of current wealth allocated to the double binary bet					
Wealth units, w_o	10^(-3)	1	10^3	10^6	10^9
Bet allocation, l_1, l_2	0.0000	0,03336	0,0499	0.05	0.05

If we are more interested in correlation structures, we could setup gambles that experience joint probabilities. For example in expression (4), by exchanging $p_1= j_1, p_2=j_2, q_3=j_3, q_4= j_4$, where j_i for i=1,2,3,4, describes a joint distribution probability matrix. In a similar fashion, it is possible to solve the analogous joint probability problem.

In this last section, we let the binary gamble approximate a financial market gamble. We introduce the random variable X with expected mean μ and variance σ^2, so that $P(X= \mu + \sigma)=p_1$ and $P(X= \mu - \sigma)=p_2$. The risk free rate earned on non-invested capital is denoted r, the EP utility function is U, and the discrete probability set is $P \in p1, p2$. The optimization function is:

$$U = p_1 U(w_o(1 + (1 - l)r + l(\mu + \sigma))) + p_2 U(w_o(1 + (1 - l)r + l(\mu - \sigma)))$$

But this gamble is of little practical use. Instead it works to illustrate the transition to the generalized gamble for N outcomes.

$$U = \sum_{i=1}^{N} p_i U\big(w_o(1 + (1 - l_i)r + l_i X_i)\big) \qquad (5)$$

,where p_i belongs to an N-vector of outcome probabilities, and X_i to a N-vector of random variable X, which consists of N pairs of μ_j and σ_j. The optimization result is shown in appendix. The utility maximization of the N-outcome problem will have to be solved implicitly for bet allocation vector l . By setting up constrains for the bet allocation, such as $l_a < l_i < l_b$, it decreases the computational work. This gamble could be interpreted as allocating wealth to a stock portfolio. The practical difficulties are, of course, to assign values for every discrete outcome. Notice that as N goes to infinity the distribution converges to a continuous probability distribution. So we have,

$$U = \sum_{j=1}^{\infty} p_i U(1 + (1 - l_i)r + l X_i) = \int_1^{\infty} U(1 + (1 - l)r + l x) dP(x)$$

, where X is a continuous random variable and dP(x) is the probability measure in the interval $0 < p < 1$. In expression (5), the gamble is described over a continuous outcome space. A conventional approach in portfolio theory is to approximate the discrete expression (5) with a continuous probability distribution in order to find a utility optimal portfolio from a large set of assets. Notice also that the binary gamble is the "exact" solution to the described portfolio problem since the large set of outcomes are finite due to trading constrains in currency denominations of gains/losses, usually hundredth of percentage, and constrains in downside and upside potential, as real asset value growth is physically constrained. This ends the result section since the discrete probability distribution has now converged to a continuous outcome space. Below follow *general discussion* and *conclusion*.

3.1. General discussion

We have argued that rationality implies that the gambler should specify gambling preferences and then decide bet allocations according to his/her utility function. This would certify that bet allocation decisions are consistent with the gambler's belief system. There are several benefits of having a specified utility functions deciding the bet allocation. It decreases the risk of subjective biases, and enables the gambler to focus on finding winning bets. But it is now clear that EP utility optimal bet allocations need to be approximated. The computational work of approximating bet allocations is fairly simple as long as gambles are single or double binary gambles. Given three coinciding gambles or more the approximation method is complex. This rational approach to gambling may only fit professional gamblers.

We analyzed a discrete approximation of a financial market gamble. It is a challenge to approximate continuous markets with a discrete probability distribution. The benefit of the discrete approximation in cases when continuous techniques are common is that the gambler is more flexible to assign probabilities to specific outcomes. With traditional portfolio theory the financial gambler faces difficulties to adjust continuous distribution for outlying event. Neglecting outliers usually coincide with asset bubbles, i.e. over-investing in a bet. In the recent financial crisis, year 2008- 2009, western financial institutions uniformly made huge losses on financial investments. This over-betting behavior may, of course, be a system failure and not necessarily inconsistent with the preferences of the financial gambler.

3. 2. Further research

This study had a strict theoretical focus by studying the gambler's decision in generalized gambles. We would be interested to see empirical studies on bet allocations decisions in relation to the gambler's a priori belief about outcome probabilities. This enables the researcher to draw inferences about gambling preferences in real world markets. Researchers may also like to evaluate if gamblers have time-consistent bet allocation strategies.

4. Conclusions

This thesis aimed to answer the specific question: How to find optimal bet allocations to a set of discrete gambles if the gambler holds an expo-power (EP) utility function.

We derived general analytical expressions that enable us to locate optimal bet allocation for specified EP-utility functions. There exist only implicit solutions to our analytical expressions. We showed how to approximate implicit solutions to specified EP-utility functions by applying the Newton-Raphson method or by using the proposed approximation formula. We analyzed the following set of discrete gambles: a single binary gamble, equivalent to betting on a tennis match, a multiple binary gamble, equivalent to betting on two tennis matches coinciding in time, and a discrete financial market gamble, equivalent to betting on an asset with finite outcome space and non-zero variance.

The main assumption in the thesis was that the gambler is von Neumann-Morgenstern rational. We used a no arbitrage argument to motivate the assumption. We further elaborated on how a VNM-utility function is characterized by the Arrow-Pratt risk aversion measure. This relationship enables us to locate interval of where to find utility optimal bet allocations and derive a general approximation formula valid if wealth approaches large numbers.

The Newton-Raphson method used to find optimal bet allocations is fairly simple for single binary gambles. As the discrete gambles grow in complexity, like betting on multiple gambles coinciding in time, the implicit solutions are increasingly harder to locate. There is an alternative approach to VNM-optimizing and that is to optimize the bet allocation so that the gambler's wealth is maximized in the long-run. This is the growth optimal portfolio (GOP) approach which was thoroughly discussed in the thesis. The growth optimal bet allocation is closely related to the VNM-optimal allocation. The growth optimal bet allocation acts as reference point to locate the VNM-optimal allocation. We argued that general VNM-preferences better capture gambling preferences than growth optimal preferences. As most people, gamblers are willing to lower the expected (geometric) mean return for lower expected variance. This leads us to the general motivation for the thesis which was to develop analytical results that enable VNM-rational gamblers to locate optimal bet allocations in discrete gambles.

As a last note, the growth optimal bet allocation for coinciding gambles need to be located with an approximation technique. So in terms of computability, the difference between the two optimization approaches, VNM and GOP, are neglectable. Professional gamblers are likely to play gambles coinciding in time and they are likely to have preferences to optimize bet allocations for discrete gambles. We advise these gamblers to not be content with the growth optimal strategy before specifying their gambling utility function.

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6. Appendix: Newton-Raphson method and analytical results

The single variable Newton-Raphson is often generalized to systems of non linear equations. Given a non-linear equation, $f(x) = 0$, the derivative of the guessed solution is approximately, $f'x_1 = \frac{\partial f(x_1)}{\partial x_1} \approx \frac{f(x_1)-f(x_2)}{x_1-x_2} \leftrightarrow f'x_1(x_1 - x_2) \approx f(x_1) - f(x_2)$. Setting $f(x_2) = 0$, the result is the iteration formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. For a non-linear system of eqautions, $f(x) = 0$, the iteration formula is: $x_{n+1} = x_n - \frac{f(x_n)}{Jf(x_n)}$, where f is a m-dimensional vector with m roots solving the system, and Jf is the Jacobian matrix. Given the existence of $\det(Jf(x)) \neq 0$, there exists a solution to the non-linear system.

Partial differentiation results to the two binary gambles coinciding in time are presented below. The results are valid for the expo power utility function, $u(x) = \theta - \exp(-\beta w^{\alpha})$, $\alpha\beta > 0$ and $\theta > 1$.

The double binary gamble was setup as:

$$U[x_1(l)] = p_1 p_2 U[w_o(1 + bl_1 + cl_2)) + q_1 q_2 U[w_o(1 - l_1 - l_2)] + p_1 q_2 U[w_o(1 + bl_1 - l_2)] + q_1 p_2 U[w_o(1 - l_1 + cl_2)].$$

The partial derivatives are,

$$U'_{l_1}(l_1, l_2) = \alpha\beta w_o[p_1 p_2 b(w_o(1 + bl_1 + cl_2))^{\alpha-1} \exp(-\beta\big(w_o(1 + bl_1 + cl_2)\big)^{\alpha}) - q_1 q_2\big(w_o(1 - l_1 - l_2)\big)^{\alpha-1} \exp(-\beta\big(w_o(1 - l_1 - l_2)\big)^{\alpha}) + p_1 q_2 b\big(w_o(1 + bl_1 - l_2)\big)^{\alpha-1} \exp(-\beta\big(w_o(1 + bl_1 - l_2)\big)^{\alpha}) - q_1 p_2 \left(w_o(1 - l_1 + cl_2)\right)^{\alpha-1} \exp(-\beta\big(w_o(1 - l_1 + cl_2)\big)^{\alpha})]$$

$$U'_{l_2}(x_1(l_1, l_2) = \alpha\beta w_o[p_1 p_2 c(w_o(1 + bl_1 + cl_2))^{\alpha-1} \exp(-\beta\big(w_o(1 + bl_1 + cl_2)\big)^{\alpha}) - q_1 q_2\big(w_o(1 - l_1 - l_2)\big)^{\alpha-1} \exp(-\beta\big(w_o(1 - l_1 - l_2)\big)^{\alpha}) - p_1 q_2\big(w_o(1 + bl_1 - l_2)\big)^{\alpha-1} \exp(-\beta\big(w_o(1 + bl_1 - l_2)\big)^{\alpha}) + q_1 p_2 c \left(w_o(1 - l_1 + cl_2)\right)^{\alpha-1} \exp(-\beta\big(w_o(1 - l_1 + cl_2)\big)^{\alpha})].$$

The derivative expressions in the Jacobian matrix are presented below.

$$x_{pp} = w_o(1 + bl_1 + cl_2; \; x_{qq} = w_o(1 - l_1 - l_2); \; x_{pq} = w_o(1 + bl_1 - l_2); \; x_{qp} = w_o(1 - l_1 + cl_2).$$

$\frac{\partial U'_{l_1}(x_1(l_1,l_2)}{\partial l_1} =$
$\alpha\beta w_0^2(p_1p_2\exp(-\beta x_{pp}^a)b^2\left[x_{pp}^{a-2}(a-1)-x_{pp}^{2a-2}a\beta\right]+q_1q_2\exp\left(-\beta x_{qq}^a\right)\left[x_{qq}^{a-2}(a-1)-\right.$
xpp2a−2aβ+ *p1q2*exp *−βxpqab2[xpqa−2a−1−xpq2a−2aβ*]+
$q_1p_2\exp\left(-\beta x_{qp}^a\right)[x_{qp}^{a-2}(a-1)-x_{qp}^{2a-2}a\beta])$

$\frac{\partial U'_{l_1}(x_1(l_1,l_2)}{\partial l_2} =$
$\alpha\beta w_0^2(p_1p_2\exp(-\beta x_{pp}^a)bc\left[x_{pp}^{a-2}(a-1)-x_{pp}^{2a-2}a\beta\right]+q_1q_2\exp\left(-\beta x_{qq}^a\right)\left[x_{qq}^{a-2}(a-1)-\right.$
$\left.x_{pp}^{2a-2}a\beta\right]-\ p_1q_2\exp\left(-\beta x_{pq}^a\right)b[x_{pq}^{a-2}(a-1)-x_{pq}^{2a-2}a\beta]-q_1p_2\exp\left(-\beta x_{qp}^a\right)c\,[x_{qp}^{a-2}(a-$
$1)-x_{qp}^{2a-2}a\beta])$

$\frac{\partial U'_{l_2}(x_1(l_1,l_2)}{\partial l_1} =$
$\alpha\beta w_0^2(p_1p_2\exp(-\beta x_{pp}^a)bc\left[x_{pp}^{a-2}(a-1)-x_{pp}^{2a-2}a\beta\right]+q_1q_2\exp\left(-\beta x_{qq}^a\right)\left[x_{qq}^{a-2}(a-1)-\right.$
xpp2a−2aβ-
$p_1q_2\exp\left(-\beta x_{pq}^a\right)b[x_{pq}^{a-2}(a-1)-x_{pq}^{2a-2}a\beta]-q_1p_2\exp\left(-\beta x_{qp}^a\right)c\,[x_{qp}^{a-2}(a-1)-$
$x_{qp}^{2a-2}a\beta])$

$\frac{\partial U'_2(x_1(l_1,l_2)}{\partial l_2} =$
$\alpha\beta w_0^2(p_1p_2\exp(-\beta x_{pp}^a)c^2\left[x_{pp}^{a-2}(a-1)-x_{pp}^{2a-2}a\beta\right]+q_1q_2\exp\left(-\beta x_{qq}^a\right)\left[x_{qq}^{a-2}(a-1)-\right.$
xpp2a−2aβ+ *p1q2*exp *−βxpqa[xpqa−2a−1−xpq2a−2aβ*]+
$q_1p_2\,c^2\exp\left(-\beta x_{qp}^a\right)[x_{qp}^{a-2}(a-1)-x_{qp}^{2a-2}a\beta])$

Analytical results for the N-outcome financial gamble are here presented. We apply the EP utility function to the gamble, and receive the expression:

$$U=\sum_{i=1}^{N}p_i(\theta-\exp\left(-\beta(w_0(1+(1-l_i)r+l_iX_i))^a\right)$$

The partial derivative w.r.t to bet allocation l_i gives us,

$$U'(l_i)=\sum_{i=1}^{N}p_i\alpha\beta w_0(-r+X_i)(w_0(1+(1-l_i)r+lX_j))^{a-1}\exp\left(-\beta\left(w_0(1+(1-l_i)r+l_iX_i)\right)^a\right)$$

Notice that marginal utility is positive only if, $X_i > r$, i.e. if $E(X_i) > r$, the expected return of the risky asset is then greater than the risk free interest rate. A utility maximum is reached at the no arbitrage condition: $E(X_i) = r$.

$$U''(l_i) = p_i a\beta w_0^2\left(-r + X_j\right)^2 \exp(-\beta(x_u)^a)\,[(x_u)^{a-2}(\alpha - 1) - a\beta(x_u)^{2a-2}]$$

$$x_u = w_0(1 + (1 - l)r + lX_j$$

The table presents approximation results for the single binary and the double binary gamble given the specified EP-utility function, in the results section. The error term is estimated as the last iteration term minus the second last iteration term.

Single binary gamble			
Wealth	% Bet allocation	Error	Marginal utility
1	0.03342	8.1185*10^(-16)	-1.1102*10^(-15)
10^3	0.05010	1.1569*10^(-6)	-1.1636*10^(-12)
10^6	0.05012	8.8114*10^(-6)	8.3676*10^(-18)
10^9	0.05012	8.7642*10^(-6)	8.7643*10^(-24)

Double binary gamble			
Wealth	% Bet allocation	Error	Marginal utility
1	0,03336	1.0839*10^(-9)	-2.220*10^(-16)
10^3	0,0499	3.4051*10^(-9)	-1.0842*10^(-19)
10^6	0.05	2.4999*10^(-8)	-1.5881*10^(-21)
10^9	0.05	9.9993*10^(-5)	-2.4552*10^(-17)