Stockholm School of Economics

B.Sc. Thesis in Finance
Spring 2013

An Equal Risk Contribution Portfolio Approach Using VIX Futures

Rickard Gunnvald
✉22344@student.hhs.se

Andreas Lagerqvist
✉22287@student.hhs.se

Tutor:
Kathryn Kaminski
September 12, 2013

Abstract

As an effect of the unusually volatile financial markets the past few years, new asset allocation strategies such as the equal risk contribution (ERC) approach has been developed. This report examines the effect on portfolio performance of adding VIX Futures contracts to an ERC portfolio consisting of positions in long-term government bonds and S&P 500 equities. The investigation is conducted through simulations using a model based on data from 2004-2008, and is supplemented by a study of the tail event in 2008-2009. The results suggest that the inclusion of VIX Futures positions substantially improves portfolio performance in terms of Sharpe ratio, and is also shown to remain significant when adjusting for transaction costs and diversification effects. Thus, investors and asset managers using the ERC approach could benefit from adding VIX Futures positions to their portfolios.

Acknowledgement

We would like to thank our tutor Kathryn Kaminski for valuable input and encouragement. Furthermore, we are thankful to Stefan Bergsten at Sarissa Asset Management and Daniel Eskilsson and Philip Hansen at J.P. Morgan for insightful comments.

Rickard Gunnvald and Andreas Lagerqvist

Contents

Chapter 1

Introduction

During the past decade, the financial markets have been unusually volatile. As a result, investors have started to rethink their asset allocation strategies. Alternative approaches based on volatility exposure, such as *risk parity* or *equal risk contribution (ERC)*, have exhibited good past performance and become increasingly popular [19]. An ERC portfolio typically consists of different asset classes, such as Developed Equity, Emerging Equity, Commodities, REITs (Real Estate Investment Trust) and Treasuries. These portfolios usually have large bond positions (or similar), and leverage is often used to increase return. Higher leverage might increase the exposure to tail risk, and during times of economic downturns or prolonged economic crises investors may find it hard to get a sufficient yield on bonds.

A natural hedge for equity portfolios is positions in volatility because of their negative correlation. One such alternative is long positions in VIX Futures, which are Futures contracts written on the VIX Index; the implied volatility of S&P 500. The index generally responds with a fast and strong upward reaction to bad market news, and Futures are used to access the movements since the index itself is not traded and there is no underlying asset or replicating portfolio. A key to capitalize on these reactions is to actively manage the position, since VIX Futures are prone to fall very sharply when market conditions improve. Furthermore, these characteristics make the VIX Index hard to predict.

An interesting question is if introducing VIX Futures as an asset class in an equal risk contribution portfolio increases the overall portfolio performance. Contributing factors could be that a well-managed VIX position has positive hedging effects and the fact that the VIX Index has a floor. In practice the floor is greater than zero, which suggests that holding a VIX position in good times does not have to give large negative returns if the position is entered at a sufficiently low level. Moreover, the underlying intuition behind the ERC approach is to maintain equal risk (volatility) shares between all assets. That is, lower the exposure to an asset class as the volatility in the asset increases and vice versa. E.g. the rapid response in VIX Futures to bad market news is characterised by increasing Futures prices and spiking volatility, and thus an ERC investor should decrease the VIX position. This might be a method to capitalize on the VIX's movements, since the relative position in VIX Futures is lowered over the whole volatility spike and invested in e.g. S&P that most likely turned down at the same time. The purpose of this paper is to investigate if ERC investors could benefit from entering a VIX Futures position using simulations.

The simulations are based on stochastic representations of the S&P 500 Index, VIX Futures contracts, long term government bonds and a roll over equal risk contribution reallocation decision rule. Using different rebalancing frequencies ranging from daily to yearly, a large sample of portfolio values over one year is generated. From this, the returns and other performance metrics such as the Sharpe ratio are calculated. To investigate VIX's impact, accessed through rolled over Futures positions, two portfolios are compared. The first consisting only of a S&P 500 position and government bonds, while the second includes VIX Futures as well. In collaboration with a calibrated model by Zhu and Lian (2012), the portfolio simulations starts at a hypothetical July 11, 2008 and terminates one year later, using simulated data. For reference, the corresponding portfolio performance using real data will be examined.

Literature on the benefits of adding VIX Futures positions to an equal risk contribution portfolio is limited, and to our knowledge this paper is the first to examine this approach. The fact that VIX Index products were introduced in 2004 and that the equal risk contribution asset allocation was first considered in the aftermath of the 2008 financial crisis might contribute to the absence of previous studies. Nevertheless, Black (2006) has shown that adding small positions in VIX

Index products would eliminate hedge funds' higher-moment risks while keeping equity returns and bond-like risks. Moreover, Maillard et al (2009) conclude that an equal risk portfolio strategy would be a good trade-off between a minimum variance portfolio and an equal weights portfolio in terms of absolute level of risk, risk budgeting and diversification.

This paper shows that adding VIX Futures to an ERC portfolio consisting of S&P 500 and government bonds substantially increases portfolio performance. Sharpe ratios based on yearly average returns increases within a range of 30 % to 128 % (corresponding to a range between 0.09 and 0.40) for rebalancing frequencies spanning from annually to daily, respectively. The results also suggest that higher Sharpe ratios can be achieved by rebalancing the porfolio including VIX Futures more frequently using the ERC allocation rule, and that VIX Futures have hedging benefits during tail events. A statistical test of the difference between Sharpe ratios for the portfolio including VIX and the portfolio without VIX shows that the results hold significantly. A further examination concludes that this still is true when effects due to transaction costs and diversification are accounted for.

As an implication, the results suggest that investors could benefit from adding VIX Futures to an ERC portfolio.

Chapter 2

Theory and Previous Studies

In this chapter, the theoretical framework and previous studies will be presented. This includes a historical review of different asset allocation decision rules, with the equal risk contribution as one of the most recent suggestions. Performance measures to be used in the portfolio evaluation in this report and some characteristics of VIX Index products as investment assets are stated. Finally, a brief mathematical background on stochastic modelling of assets is conferred.

2.1 Asset Allocation

There is a wide consensus in the financial markets that strategic asset allocation is important for overall portfolio performance. Asset allocation is based on the fundamental idea that different types of asset classes perform differently across different types of economic and market conditions. Given that asset classes are not perfectly correlated, the investor can diversify a portfolio to remove asset-specific/idiosyncratic risk. The asset allocation should also take into account an investor's specific risk preferences, time horizon or other goals.

2.1.1 Equal Weights

This asset allocation strategy is perhaps the most basic one and is often termed "the 1/N heuristic". The investor simply puts equal capital weights on the different assets and rebalances when appropriate (given transaction costs etc.). On the positive side, it is so simple that almost anyone can do it. The difficult part for an ordinary investor might be finding a suitable rebalancing rule/frequency. On the negative side, this method has not taken into account any information regarding the individual assets' characteristics; return, volatility and correlation with other assets. This means that the portfolio probably is not well-diversified and therefore the investor takes on risk that he or she is not paid for.

2.1.2 Minimum Variance Portfolio

The Minimum Variance Portfolio approach was introduced by Harry Markovitz in a famous article published 1952 in the Journal of Finance. Using historical data, the average returns and the covariances between the assets are calculated, and the Efficient Frontier is constructed by fixing an expected return under the condition that variance should be minimized. Given a risk-free return the investor can choose portfolios along the Capital Allocation Line at any desired level of risk.

An advantage of this method is that it accounts for the individual assets' mutual correlations and assigns weights to minimize portfolio variance, and hence risk. On the negative side, investment decisions are taken based on historical data, and past performance is not a guarantee for similar future results.

2.1.3 Most Diversified Portfolio

The Most Diversified Portfolio (MDP) was introduced by Choueifety and Coignard (2008) and builds upon a measure called the Diversification Ratio (DR). The DR is defined as the ratio of the portfolio's weighted average volatility to the portfolio's overall volatility. Since assets are not perfectly correlated, this ratio will typically be greater than one. Thus, the DR measures the diversification gained by holding non-perfectly correlated assets. By maximizing this measure, the Most Diversified Portfolio is obtained. Maximizing the measure implies that the individual assets have as high volatility but as low correlation as possible, conditional on the maximization of the quotient.

The measure experiences critique since the authors want to maximize the distance between two different volatility measures of the same portfolio. As Lee (2010) points out, the weighted average volatility of the portfolio is not a measure that really exists in the real world, whilst the portfolio's overall volatility clearly is a real world property. More specific, the weighted average implies that there is no diversification at all, which of course is not true in reality. The lack of an underlying economic theory, or an underlying utility function of the DR, makes it a bit unclear to why investors should maximize this measure. Also, for the MDP to end up on the efficient frontier, all assets must have identical Sharpe ratios. The assets are typically not required to be identically correlated, but the lack of this assumption at the efficient frontier implies arbitrage opportunities. However, when assets are identically correlated, it can be shown that the MDP and ERC portfolios are the same [14].

2.1.4 Equal Risk Contribution

In a Risk Contribution (RC) portfolio an investor determines asset class weights based on their contribution to portfolio risk (volatility). In the special case where the investor wants the assets to contribute equally much to the portfolio risk, an asset allocation approach known as risk parity or equal risk contribution (ERC) is obtained. Equal risk contribution has over the past decade gained attention since it has shown to outperform several classical types of asset allocation strategies [19].

One of the reasons that the ERC approach emerged was that in the standard institutional portfolio consisting of 60 % equity and 40 % bonds/fixed income, the equity part stands for over 90 % of the risk [19]. This, combined with the fact that e.g. a minimum variance approach is based on forecasts of future returns from historical data give rise to a lack of robustness that the ERC approach seems to mitigate [16]. However, the underlying economic theory supporting an ERC approach is somewhat vague. E.g. that all assets must be identically correlated and have identical Sharpe ratios for the portfolio to end up on the efficient frontier [14].

Recent Views on ERC

As the equal risk contribution approach has become increasingly popular and caught the attention of both academics and the financial industry, some studies have explored it in further detail. Romahi and Santiago (2012) have investigated data in various time periods from 1927 to the present day, and suggest that a factor premium risk parity approach outperforms the traditional ERC approach. The idea behind the risk factor approach is to use risk factors (and their premia) instead of asset classes defined in a more traditional way (commodities, equities etc.). The factors are e.g. *small cap premium*, *momentum*, *REITs* and *merger arbitrage*. One concern with traditional equal risk contribution is the increasing correlation between asset classes. Remember that for the ERC portfolio to be on the efficient frontier, one would like the asset correlation to be zero (or identical). Romahi and Santiago (2012) show that the rolling correlation between risk factors are much lower (less than 25%) when compared to traditional asset classes (about 80%), mitigating this concern. A concern regarding the risk factor approach is that some of the risk factors might be difficult to capture and/or illiquid for investors to access. Another problem is that much of the benefits are lost if the investor does not have mandate to use leverage or short sell. Nevertheless, according to Romahi and Santiago, the market is moving in a direction that makes the risk factors more liquid, and that a long-only risk factor approach still would be more beneficial than a traditional equal risk contribution approach [19].

As mentioned above, equal risk contribution is a special case of a risk contribution portfolio. Lee (2010) sees it as a starting point for an investor without a clear view of the markets. This approach is also taken further in the article by Rappoport and Nottebohm (2012). Their conclusion is that an equal risk contribution approach has performed well during the recent periods of great uncertainty, but that equal risk shares is meaningless as an objective in itself. They argue that if an investor *knows* that a certain asset will have a return of 8 %, it would be wise not to ignore it. But if the investor is a bit uncertain, then he or she might tone down his or her reliance on this forecast. Given that an investor is very uncertain about the forcast, it might be wise to ignore it. Rappoport and Nottebohm (2012) argue that this is exactly what the risk parity/equal risk contribution approach does, and hence consider it to be a starting point, just as Lee (2010).

The question is of course whether an ERC portfolio will be profitable in the future. To take into account the investor's level of uncertainty in the asset weight assessment, Rappoport and Nottebohm (2012) suggest a "Forecast Uncertainty Hedge" rule. The method "is a hybrid of a number of asset allocation proposals that have appeared in the finance literature over the last twenty years" [18] and is out of scope for this report. However, according to their simulations, this extension of equal risk contribution has outperformed traditional ERC in 70 % of the cases.

2.2 Performance Statistics

By rational economic reasoning, it is obvious that investors seek high returns on investments and simultaneously prefer to take on an as low risk as possible. In basic portfolio theory, the risk is often measured as the volatility (standard deviation) of the portfolio returns, and a fundamental relationship is the trade off between return and volatility. Already in 1966, William Sharpe presented a theoretical measure that relates the excess return with its volatility. This was later called the Sharpe ratio (SR) and is defined by:

$$SR = \frac{E[R] - r_f}{\sqrt{Var(R - r_f)}} \tag{2.1}$$

where R is the return of the portfolio and r_f the risk-free rate. In this report, r_f is assumed to be constant and hence vanishes from the denominator. One of the objectives of this report is to measure the performance of different investment portfolios in terms of the Sharpe ratio. To be able to make appropriate statistical inferences, further assumptions has to be made on the distribution of the returns. Following the approach by Lo (2002), assuming independent and identically distributed (i.i.d) returns (which would be the case for the returns of the simulated portfolios in this report), one can show that the standard errors (SE) of the estimates of the SR is given by:

$$SE(\widehat{SR}) = \sqrt{\left(1 + \frac{1}{2}\widehat{SR}^2\right)} \tag{2.2}$$

The i.i.d. assumptions gives an asymptotic normal distribution, which could be used to test for significant differences in Sharpe ratios. Moreover, for completeness and as investigation of the tails of the distribution of returns, worst drawdown and the highest return is also considered as performance statistics.

2.3 VIX and VIX Futures

On the Chicago Board Option Exchange, contracts on the Volatility Index (ticker: VIX) have been traded since March 26, 2004. The VIX is based on averaging weighted prices on put and call options traded on the S&P 500 Index for a large range of strike prices, see Appendix C.2 for the exact calculation. Therefore, the VIX Index is the level of volatility implied by option prices and is regarded as a measure of expected future market volatility. Thus, taking VIX positions is often regarded as a pure volatility exposure on the S&P 500 Index.

Historically, VIX is negatively correlated with the S&P 500 Index (correlation coefficient about -0.8). This suggests a diversification benefit when including VIX products in an investment portfolio [5]. A study performed by Black (2006) suggests that an inclusion of a small position in VIX Index products as a part of a hedge fund investment portfolio would be preferable. It would eliminate some parts of the higher-moment risks (e.g. skewness and kurtosis) while keeping a desirable portfolio with equity returns and bond-like risks. One objective of this report is to investigate if this, in some manner (e.g. in terms of Sharpe ratio), could be transferred to hold for equity and bond positions in an equal risk contribution portfolio.

One concern often stated when investing in VIX Index products is that there are no underlying assets traded. Consequently, one could not replicate the performance of the index. Therefore, contracts written on the VIX Index are only settled with cash flows at time of maturity. Futures contracts are traded in terms of the volatility (not in $) and a change of a 0.01 tick changes the balance on the margin account with $10 [4]. The initial margin for a contract is $5,500. VIX Futures contracts are often characterized by the property that contracts with longer maturity time are more expensive.

2.4 Mathematical Theory

In this section, the crucial mathematical properties of stochastic processes used in the simulations in this report are presented. The theory regarding Wiener Processes and the Geometric Brownian Motion is the most fundamental building blocks to get the stochastic representations needed for simulations (and pricing). For more technical details on arbitrage-free pricing see Appendix C.

2.4.1 Stochastic Processes

Stochastic processes are fundamental in mathematical finance since they can be used to model e.g. bonds and stocks. Representations of stochastic processes is also required to be able to analytically calculate arbitrage-free prices of financial derivatives (contingent claims). The classical example is the mathematical derivation of the well-known Black-Scholes-Merton formula. In this theory, asset prices are described as *continuous time* stochastic processes, using *diffusion processes* and *stochastic differential equations* (SDEs).

A stochastic process X is a *diffusion* if its local dynamics can be approximated by a stochastic representation of the following type:

$$\begin{cases} X(t+\Delta t) - X(t) = \mu(t, X(t))\Delta t + \sigma(t, X(t))(W(t+\Delta t) - W(t)) \\ X(0) = x_0 \end{cases}$$

where x_0 is a constant, μ and σ are deterministic functions and W is a Wiener process. Making the increments infinitesimal, i.e.letting $\Delta t \to dt$, yields the SDE:

$$\begin{cases} dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW(t) \\ X(0) = x_0 \end{cases}$$

where μ is called the *drift* term and σ is called the *diffusion* term.

Wiener Processes

The choice of Wiener processes as the driving random component in many SDEs is based on the good properties for analytical calculations, inherited from the Normal distribution. Informally, a Wiener process's non-overlaping increments are independent. This is of course a convenient when it is desirable to add pure randomness to some other process. As seen in condition 3 below, every time increment of a Wiener process has a normal distribution, with expected value zero and variance linearly increasing with the length of the time interval. It is reasonable that a large sample of processes that essentially are "random walks" will have an increased variance when the time considered increases. Formally, a stochastic process W is called a Wiener process if the following conditions hold:

1. $W(0) = 0$
2. The process W has independent increments, i.e if $r < s \leq t < u$, then $W(u) - W(t)$ and $W(s) - W(r)$ are independent stochastic variables.
3. For $s < t$ the stochastic variable $W(t) - W(s)$ has a Normal distribution $N(0, t - s)$.
4. W has continuous trajectories. (P-a.s.)

Note: The notation $N(\mu, \sigma^2)$ refers to mean μ and variance σ^2. P-a.s. means *almost surely* convergent.

One trajectory of a Wiener process is visualized below:

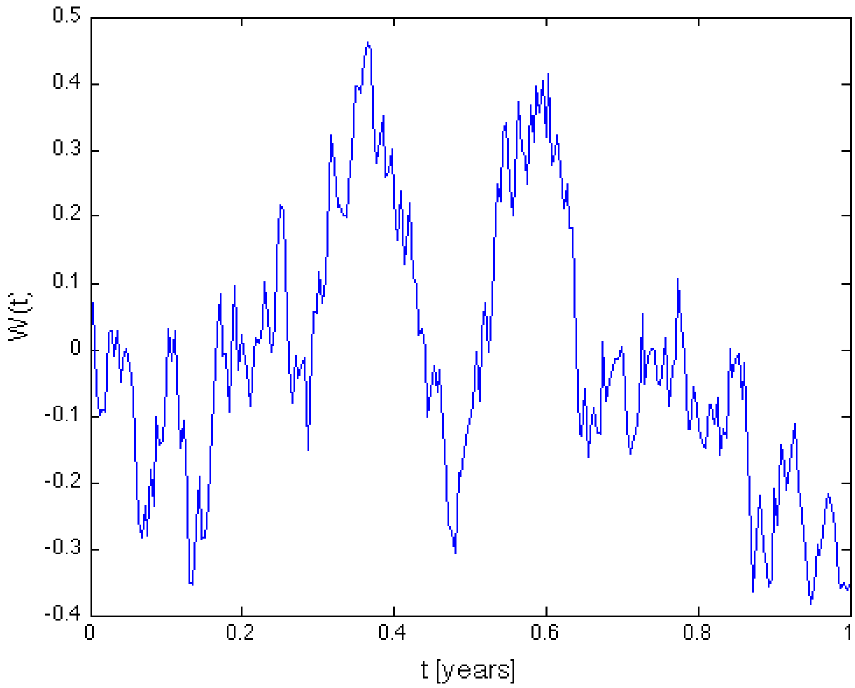


Figure 2.1: Graph of a Wiener process with dynamics $\sigma dW(t)$ with $\sigma = 0.1$ over 1 year.

Geometric Brownian Motion

Another fundamental part of asset pricing and modelling is stochastic processes that follow a geometric Brownian motion (GBM). The GBM is a type of SDE and has the dynamics:

$$\begin{cases} dX_t = \alpha_t X_t dt + \sigma_t X_t dW_t \\ X(0) = x_0 \end{cases}$$

The solution to this stochastic differential equation is given by (see [1]):

$$X_t = x_0 \cdot \exp\left\{ \left(\alpha_t - \frac{1}{2}\sigma_t^2 \right)(t - 0) + \sigma_t (W_t - W_0) \right\} \tag{2.3}$$

Using the properties of Wiener processes, we know that

$$Z_t = (W_t - W_0) \in N(0, t - 0)$$

hence the GBM follows a log-normal distribution which is one of the key assumptions in the famous Black-Scholes-Merton pricing formula. One simulation of a stock price given by a GBM is presented in Figure 2.2.

In the simulations in this report, a GBM-approach with both stochastic drift and stochastic diffusion term will be used. For more information, see Ch. 3.

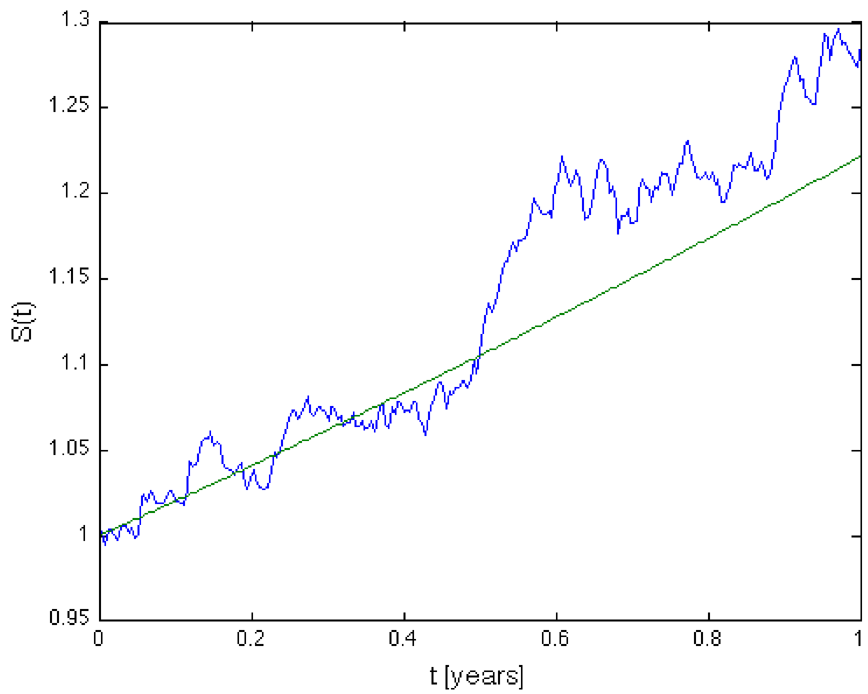


Figure 2.2: Graph of a geometric Brownian motion with dynamics $dS_t = \alpha S_t dt + \sigma S_t dW_t$, where $\alpha = 0.2$ and $\sigma = 0.1$ over 1 year. The green line is the solution to the corresponding ODE with dynamics $dS_t = \alpha S_t dt$.

Chapter 3

Methodology

The approach used to investigate equal risk contribution portfolios' performance is based on simulations and statistical examination of the generated outcomes. To simulate the different assets that are included in the portfolios, a joint model of the assets' dynamics is needed. The building blocks used are stochastic representations of S&P 500 equities, VIX Futures contracts and long term government bonds. Simulations are made using the stochastic representations together with a rebalancing rule that allocates capital in accordance with the ERC theory.

For different rebalancing frequencies, yearly returns of the portfolios are generated and the performance statistics presented in Sec. 2.2 are calculated. The models used to simulate these investment assets will be presented in this chapter. Moreover, necessary limitations and simplifications are stated as well as the technicalities of asset allocation and rebalancing. Lastly, a method to investigate the performance of these portfolios on actual data will be considered.

3.1 Stochastic Modelling

To allow for an integrated simulation of the assets in the equal risk contribution approach a comprehensive model for the asset prices is needed. The model for stocks and VIX Futures used is a simplified version of the more general presentation in Zhu and Lian (2012). As in their approach, let the stocks S_t describe the S&P 500 Index and V_t be the stochastic diffusion component. Under the objective probability measure P these are assumed to have the following dynamics:

$$\begin{cases} dS_t = S_t(r_t + \gamma)dt + S_t\sqrt{V_t}dW_t^S, & S_0 = s_0 \\ dV_t = \kappa(\theta - V_t)dt + \sigma_V\sqrt{V_t}dW_t^V, & V_0 = v_0 \end{cases} \tag{3.1}$$

where r_t is the spot interest rate, γ the equity premium and κ, θ and σ_V are, respectively, the mean-reverting speed parameter, the long-term mean and the variance coefficient of the diffusion V_t, and dW_t^S and dW_t^V are two standard Wiener processes under P with correlation ρ. This implies that the S&P 500 Index is described as a geometric Brownian motion with a stochastic diffusion term, and that the diffusion term is a mean-reverting stochastic process. Mean-reverting meaning that the diffusion term fluctuates around, in this case, θ. Following the standard approach in arbitrage theory, the dynamics could be expressed under the risk-neutral martingale measure Q as:

$$\begin{cases} dS_t = S_t r_t dt + S_t\sqrt{V_t}dW_{t,Q}^S, & S_0 = s_0 \\ dV_t = \kappa_Q(\theta_Q - V_t)dt + \sigma_V\sqrt{V_t}dW_{t,Q}^V, & V_0 = v_0 \end{cases} \tag{3.2}$$

where $\eta_V = \kappa_Q - \kappa$ is the volatility risk premium and σ_V, ρ and $\kappa\theta$ are preserved under the (Girsanov) measure transformation from P to Q. From this, Duan and Yeh (2007) have shown that an explicit formula for the VIX Index is given by:

$$VIX_t^2 = (aV_t + b) \tag{3.3}$$

where

$$\begin{cases} a = \dfrac{1 - e^{-\kappa_Q\bar{\tau}}}{\kappa_Q\bar{\tau}}, \bar{\tau} = 30/365 \\ b = \theta_Q(1 - a) \end{cases} \tag{3.4}$$

Using this and the standard approach for pricing Futures contracts, Zhu and Lian (2012) have shown that the VIX Futures price at time t is given by:

$$F(t, T, VIX_t) = E^Q[VIX_T|\mathcal{F}_t] = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{1 - e^{-sb}f\left(-sa; t, \tau, \dfrac{VIX_t^2 - b}{a}\right)}{s^{1.5}}ds \tag{3.5}$$

where T is the maturity time of the VIX Futures contract, $\tau = T - t$ time to maturity and:

$$f(\phi; t, \tau, V_t) = e^{C(\phi,\tau)+D(\phi,\tau)V_t} \tag{3.6}$$

with

$$\begin{cases} C(\phi,\tau) = \dfrac{-2\kappa\theta}{\sigma_V^2} \ln\left(1 + \dfrac{\sigma_V^2\phi}{2\kappa_Q}(e^{-\kappa_Q\tau} - 1)\right) \\ \\ D(\phi,\tau) = \dfrac{2\kappa_Q\phi}{\sigma_V^2\phi + (2\kappa_Q - \sigma_V^2\phi)e^{\kappa_Q\tau}} \end{cases} \tag{3.7}$$

Hence, solving the system of stochastic differential equations (3.1) and calculating the integral (3.5) gives the values for the S&P 500 and Futures prices on VIX.

Furthermore, to be able to use this approach a specification of the short rate r_t is needed. In the literature, there exists a large number of different short rate models. A fairly general alternative is the Hull-White model (3.8), which fits theoretical bond prices to the observed yield curve. The dynamics under the risk neutral probability measure Q is given by:

$$dr_t = (\Theta(t) - \tilde{a}r_t) + \sigma dW_{t,Q}^r, \quad r_0 = \tilde{r}_0 \tag{3.8}$$

where $\Theta(t)$ is a deterministic function of time to be specified by the observed forward yield curve, $\tilde{a}$ and σ are constants and W_t^r is a Q-Wiener process (at this point with no restrictions).

Hull and White (1990) have shown that this model yields the zero coupon bond prices $p(t,T)$ at time t maturing at time T as:

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)} \exp\left[B(t,T)f^*(0,t) - \frac{\sigma^2}{4\tilde{a}}B^2(t,T)(1 - e^{-2\tilde{a}t}) - B(t,T)r(t)\right] \tag{3.9}$$

where $f^*(0,t)$ is the observed t-forward rate at time zero, $p^*(0,t)$ and $p^*(0,T)$ the observed zero coupon bond prices at time zero with maturity time t and T respectively, and with:

$$B(t,T) = \frac{1}{\tilde{a}}\left[1 - e^{-\tilde{a}(T-t)}\right] \tag{3.10}$$

Using the notation in Björk (2009), it can be shown that:

$$\Theta(t) = f_T^*(0,t) + g'(t) + \tilde{a}\left(f^*(0,t) + g(t)\right) \tag{3.11}$$

where the subscript T indicates partial derivative with respect to the time to maturity and:

$$g(t) = \frac{\sigma^2}{2} B^2(0,t) \tag{3.12}$$

Moreover, a useful relationship for calculating bond prices from the forward rate curve is given by:

$$f^*(0,T) = -\frac{\partial \ln(p^*(0,T))}{\partial T} \tag{3.13}$$

The consequences of adjoining this model to the specification in (3.1) will be examined in Sec. 3.2.

Consequently, the approach chosen to model stocks, bonds and VIX Futures prices consists of the solution to the stochastic differential equation (3.1) for S&P 500, the values of the integrals in (3.5) and the bond prices given by (3.9). The model uses the parameters: γ, s_0, κ, θ, σ_V, v_0, η_V,$\bar{\tau}$, $\tilde{a}$, σ, $\tilde{r}_0$ and the yield curve $f^*(0,T)$ to determine the variables: S_t, $F(t,T,VIX_t)$ and $p(t,T)$ as functions of t and T.

3.2 Limitations and Assumptions

The model for the S&P 500 Index and the Futures prices on VIX found in Zhu and Lian (2012) used and presented in this report is already a simplification of the most general model presented in their paper using e.g. stochastic jump processes in both the underlying and the diffusion term. A more general model is out of scope for this paper, and the simplification is justified by Zhu and Lian's conclusion that this model yields a sufficient and reasonable fit with the historical data on VIX Futures prices. The model in this report limits the VIX Index modelling, which in practice is characterized by volatility spikes which often are modelled by jump processes. However, since the equal risk contribution portfolio consists of Futures on the VIX Index rather than VIX Index positions directly, the model serves the desired purpose of usage.

Further simplifications used in this report regard model parameter estimations. As suggested and performed by Zhu and Lian (2012), the solution to the system of stochastic differential equations (3.1), (3.3) and Futures prices (3.5) is fitted to historical data on the S&P 500 and the VIX Index using Markov chain Monte-Carlo simulation and parameter estimation. Since the scope of this report is focused on asset allocation in portfolio management rather than mathematical model estimation, the estimates found in Zhu and Lian (2012) is used and shown in Table 3.1. These parameters are estimated from daily close levels on S&P 500 and VIX as well as daily VIX Futures settlement prices over the period March 26, 2004 to July 11, 2008. The date March 26, 2004 is when the trading with VIX Futures started and July 11, 2008 is presumably chosen as to match with the time frame of their research.

Moreover exclusion filters are used, e.g. removal of Futures with less than five days to maturity and Futures with less than 200 contracts traded to avoid liquidity-related biases. One implication for this report of choosing the parameters in Table 3.1 rather than up-to-date estimations is that the results of the simulations regarding rebalancing frequency can be compared to the actual market performance for the time period July 12, 2008 and onwards. Moreover, choosing this period of time for parameter estimation seems rather beneficial in the sense that the markets were fairly stable during this period, as could be seen in Figure 3.1. I.e. the conclusions made from these simulations could be said to hold in a more general setting than if post 2008 financial crisis parameters are used, since these would be biased on a market suffering from distress. In addition, using parameters from a stable market period to examine the performance using actual data from a crisis would give an indication on the portfolio performance in tail events. This is further examined in Section 3.4.

Table 3.1: Estimates of model parameters from Zhu and Lian (2012). The table reports annualized means and standard deviations within parentheses for each estimator.

θ	κ	σ_V	η_V	ρ
0.0444	2.2680	0.3856	-2.0160	-0.7533
(0.0071)	(0.2520)	(0.0504)	(0.5040)	(0.0231)

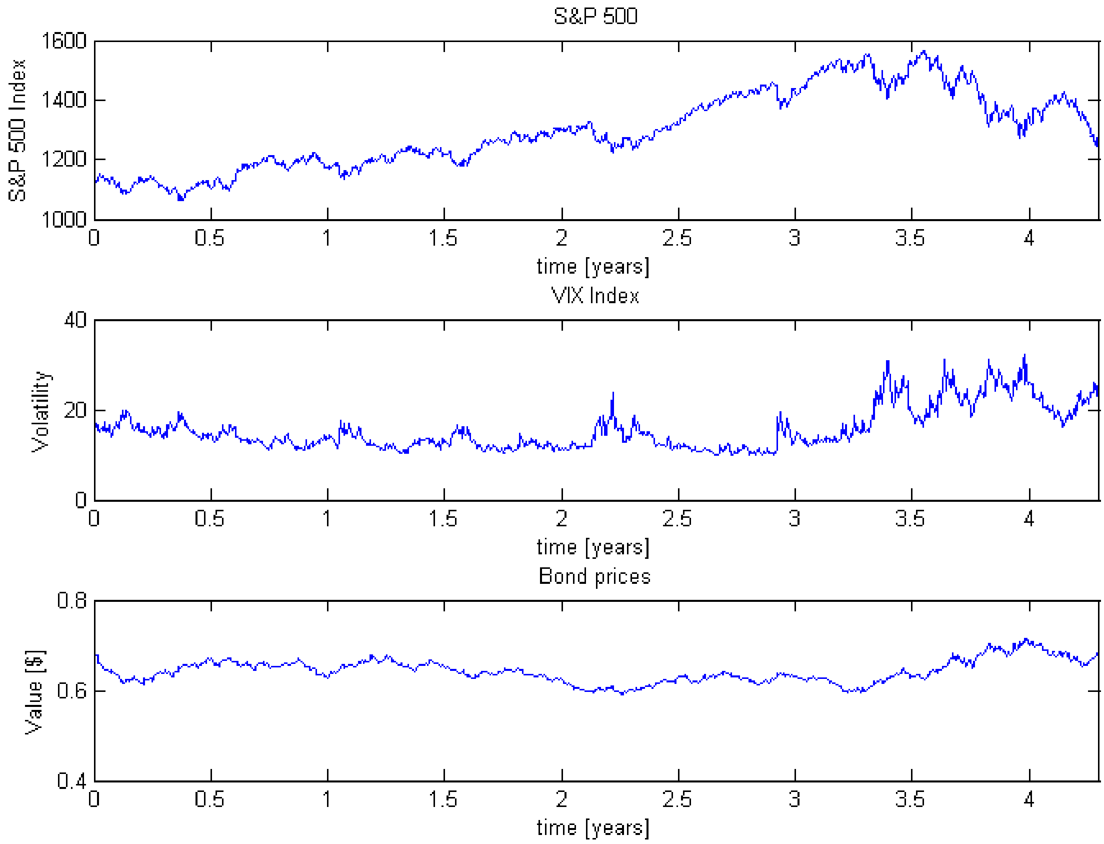


Figure 3.1: Historical data on S&P 500, VIX Index and 10 year government zero coupon bonds from March 26, 2004 to July 11, 2008.

Extending the model used by Zhu and Lian (2012) with the Hull-White short rate model in a general setting generates a model examined by Grzelak, Ooseterlee and van Weeren (1990), with correlations among all the driving Wiener processes. In that approach, it can be shown that there does not exist affine solutions, which means that e.g. the bond prices could not be calculated using equation (3.9), and more sophisticated pricing formulas are necessary. In the scope of the simulations in this report, and as a first approximation, an affine term structure is considered. The consequence of this simplification is that the short rate will be modelled independently of the processes described in (3.2). One should also note that the bond positions are assumed to be in terms of zero coupon bonds with face value \$1. The zero coupon bond is a theoretical construction that is not traded in the bond market. However, the zero coupon bond yields could easily be derived from actual government (coupon) bonds and vice versa. Zero coupon bonds are thus used in this report for transparency.

Moreover, a corresponding simplification regarding parameters estimation for the short rate model is used. If an explicit calibration is to be made, either caplet or swaption prices could be used. The estimates to be used in this report are values that are frequently used in simulations in the literature, e.g. by Schulmerich (2010), and are shown in Table 3.2.

Table 3.2: Estimates of short rate model parameters by Schulmerich (2010). The table reports annualized means.

σ	$\tilde{a}$
0.01	0.02

The initial parameters and the equity premium have to be stated or estimated. The initial values s_0, v_0 and $\tilde{r}_0$ are taken from S&P 500 Index value at July 11, 2008, Zhu and Lian's report and U.S. Department of the Treasury yield curve from July 11, 2008 (by linear extrapolation) respectively. The equity premium γ is taken from the research of Graham and Harvey (2008). Moreover, VIX_0 is adjusted to the data above. All values are shown in Table 3.3, and the U.S. Department of the Treasury forward rates on July 11, 2008 are shown in Table 3.4.

Table 3.3: Initial parameters and the equity risk premium γ. The table reports annualized values.

s_0	v_0	$\tilde{r}_0$	γ	VIX_0
1239.49	0.0225	0.0126	0.037	0.22

Table 3.4: Forward rates quoted by the U.S. Department of the Treasury on July 11, 2008.

Maturity	1 m	3 m	6 m	1 y	2 y	3 y	5 y	7 y	10 y
	1.38%	1.62%	2.02%	2.30%	2.59%	2.88%	3.27%	3.55%	3.96%

Furthermore, the parameters are assumed to be fixed during the simulations. In practice, an ERC investor would update the parameters gradually as time passes by. This would require parameter estimation and is thus out of the scope of this report. Moreover, the simulations are on a one year horizon (252 trading days) due to the non-updated parameters, since longer time periods would require higher model and estimation accuracy.

The time to maturity of the Futures contracts and the bonds simulated in this report uses a roll over simplification, i.e. the Futures contracts to be traded are assumed to always have a maturity time of three months and the long-term government bonds are equally assumed to have a maturity of 10 years. A maturity time of three months for the Futures is used since a short maturity is preferred due to higher VIX Index dependence in pricing, and a shorter maturity time would not be transferable to real life investment decisions since Futures with shorter maturities are also experiencing liquidity related effects. In practice, the settlement of the VIX Futures is thirty days prior to the maturity of S&P 500 options. Thus, the assumption of Futures with maturity time of three months gives a difference in time to maturity compared to real Futures of at most two weeks, and on average no difference.

Long term government bonds are used to match the return profile of an equal risk contribution investor, i.e. increase both expected returns and risk exposure. Furthermore, some simplifications regarding the margin account for the

Futures contracts are used. There is no interest gained or paid and the investor does not get any margin calls. The implication is that there is a little less return in the simulated portfolio including VIX Futures than what it should be. This is because either a small return is given due to interest on the margin account, or that the investor believes that a higher return could be achieved by using leverage and investing the "margin account money" elsewhere. In practice, the terms of these accounts vary among investors, and it is easily adapted to the model if desired.

3.2.1 Turnover and Transaction Costs

Another simplification used is regarding transaction costs, which at a first approximation are ignored. The reason is mainly for simplicity of the results presented in this report. Moreover, the transaction costs vary over time and is different among investors. An investor interested in this approach could add his or her transaction costs into the simulation.

For robustness, a review of the difference in transaction costs between portfolios with and without VIX Futures positions, the total net reallocation of each simulation is calculated and shown (in average for each rebalancing frequency) in Ch. 4. The definition of net total reallocation (turnover) used is the sum of the absolute change in each portfolio weight for each reallocation over the whole year of investment. From this, an estimate of the effect on portfolio performance from the total aggregated transaction costs can be calculated. In Ch. 4, a transaction cost of 0.05 percent is assumed, and the impact on the different portfolios is calculated. This does not take care of the effect of continuously settling the transaction costs and thus the effect of compounding, which for these purposes can be considered to be negligible.

3.3 Simulation in Matlab

The Matlab-simulation implements the models stated in Sec. 3.1 with parameters from Sec. 3.2 using time-discretization with equal time-steps corresponding to daily values. The Wiener increments are normally distributed, with W^S and W^V

correlated by ρ and independent of W^r. Modelling is preferably performed under the objective measure P, but for uniqueness the short rate model must be simulated under the risk-neutral measure Q. Note that the trajectories of r used in e.g. (3.1) is the risk-neutral values, hence the modelling will be consistent. The log-return of the S&P 500 is modelled in analogue with Zhu and Lian (2012), and denoted $Y_t = \ln(S_t/S_{t-1})$. The term arising from the diffusion could be neglected when applying Itô's formula.

The discretization used with time-increments Δt is given by:

$$\begin{cases} Y_{t+1} = (r_t + \gamma)\Delta t + \sqrt{V_t}\sqrt{\Delta t} Z_t^S, & S_0 = s_0 \\ V_{t+1} = V_t + \kappa(\theta - V_t)\Delta t + \sigma_V \sqrt{V_t}\sqrt{\Delta t} Z_t^V, & V_0 = v_0 \\ r_{t+1} = r_t + (\Theta(t) - \tilde{a} r_t)\,\Delta t + \sigma\sqrt{\Delta t} Z_t^r, & r_0 = \tilde{r}_0 \end{cases} \tag{3.14}$$

where Z_t^S, Z_t^V and Z_t^r are standard normal distributed and Z_t^S and Z_t^V have correlation ρ. Subsequently, the Futures prices are calculated using the numerical integration function `quadgk` and the bond prices are calculated using (3.9). Moreover, a total of 1000 simulations of each trajectory is used. The number 1000 is chosen as a trade-off between precision (size of standard errors of estimates), needed computational power and for easy reference. As could be seen in Ch. 4, the precision is satisfactory for the purposes of this report. I.e. the precision is enough to conclude that the estimates of the portfolios' Sharpe ratios are significantly different when including VIX Futures positions.

3.3.1 Allocation of Capital

With the different trajectories simulated for the S&P 500, the VIX Futures and the bonds, the asset allocation will be according to an equal risk contribution approach. Using the methodology presented by Maillard et. al. (2009), the capital will be allocated based on volatilities and correlations as risk measures. This gives an allocation rule for the weights $w_{S\&P}$, w_{VIX} and w_{bond} in form of a minimization problem. The desired allocation is such that $w_i(\Sigma \cdot w)_i = w_j(\Sigma \cdot w)_j$ for all $i, j = \text{S\&P}, \text{VIX}, \text{bond}$ and where Σ is the covariance matrix between the assets' returns. This implies that all assets contribute equally to the total portfolio risk.

The optimization problem to be solved in order to get portfolio weights could be written as:

$$\begin{aligned} &\min \textstyle\sum_i \sum_j (w_i(\Sigma \cdot w)_i - w_j(\Sigma \cdot w)_j)^2, \quad i, j = \text{S\&P}, \text{VIX}, \text{bond} \\ &\text{subject to } \textstyle\sum_i w_i = 1, w_i \geq 0 \end{aligned} \tag{3.15}$$

where $w = (w_{S\&P}, w_{VIX}, w_{bond})^T$ represents the vector of portfolio weights. The optimization constraints is interpreted as that no short-selling is allowed, that all the weights should sum up to one means that the portfolio is unlevered. By inspection, the objective function equals zero only if the desired allocation is fulfilled. The covariance matrix Σ has to be estimated using historical data, which in the context of this report is done by using previous values in the simulated trajectories. The chosen backward time is set to seven days, as a trade-off between sample-size and importance of historic events. A larger sample will give a higher accuracy on the estimates, while the importance of past events will be present for a longer time when more data points are used.

According to Maillard et. al. (2009), the intuitive allocation problem presented in equation (3.15) could be rewritten as a convex optimization problem with a *unique* solution, which is why the rewritten version is used in the simulations made in this report. The optimization problem could then be formulated as:

$$\begin{aligned} &\min \sqrt{y^T \Sigma y} \\ &\text{subject to } \textstyle\sum_i \ln(y_i) \geq 0, y_i \geq 0 \end{aligned} \tag{3.16}$$

where y is the un-normalized analogue of w. Consequently, the portfolio weights are given by:

$$w_i = \frac{y_i}{\sum_i y_i} \text{ where } i = \text{S\&P}, \text{VIX}, \text{bond} \tag{3.17}$$

Furthermore, the initial capital is assumed to be $\$1,000,000$ and given the weights of the allocation, the number of positions in each asset can be determined. The size of initial investment is irrelevant for this study, but is important if real costs (such as a lower bound in dollars of a transaction cost) are implemented. The initial value is easily changed to a chosen number in the MATLAB-code. For S&P 500 and the government bonds, the number of positions is simply determined as the value of total allocation divided by the price of one position in S&P 500 and in bonds respectively. For the Futures contracts, the allocation is determined in terms of the initial margin for one contract. In particular, the number of Futures contracts held is the amount allocated to VIX Futures divided by the initial margin. This rule is plausible since it would not require any further assumptions on the terms of the margin account, as described in Sec. 3.2.

3.3.2 Rebalancing Decision Rule

The rule for rebalancing is simply a fixed rebalancing frequency, based on a roll over strategy. With $\widehat{\Sigma}$ estimated from returns over the past seven time points, equation (3.16) is used to assign weights between the asset classes in the portfolio. In this report daily, weekly, monthly, quarterly, semi-annually and annually rebalancing frequencies are investigated. A roll over strategy means that for each time of rebalancing, an updated covariance matrix is used to calculate the new weights. This yields that a calibration of the model is needed for the first seven days, which is realized by starting the simulations seven days before the capital is firstly allocated.

3.4 Performance on Historical Data

Using the same approach as in Sec. 3.3.2 and Sec. 3.3.1, the performance of the equal risk contribution portfolio could be examined on historical data from July 11, 2008 to July 10, 2009. The data used is the actual values of the S&P 500 Index, the zero coupon bond prices calculated from the forward rate curve and VIX Futures with 2.5 to 3.5 months to maturity (uniformly distributed). Note that since it is assumed that there always exists Futures maturing in three months in the simulations, the maturity for the actual data is chosen as close to three months as possible. The realizations could be seen in Figure 3.2. Another note on the performance on historical data is that there is only one realization, hence the only measure that is meaningful is the returns of the different portfolios. E.g. the Sharpe ratio would be undefined since the variance of one observation is zero.

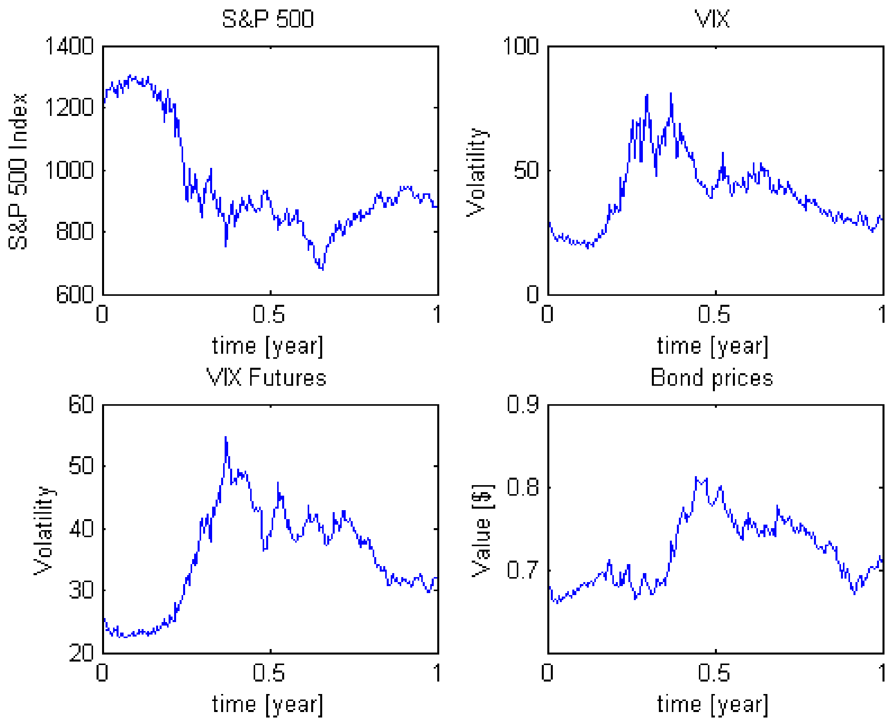


Figure 3.2: Graphs showing the actual outcome of S&P 500, VIX Index, 3-month VIX Futures and 10-year U.S. government zero coupon bond price during the time period 11 July, 2008 - 10 July, 2009.

3.5 Portfolio Performance

Given the simulations of the portfolios with and without VIX Futures positions for different rebalancing frequencies, the performance statistics presented in Sec. 2.2 are calculated. The simulation generates the yearly return of each of the 1000 portfolios with and without VIX Futures, respectively. The arithmetical mean is used to estimate the average yearly return and the square root of the sample variance estimates the annualized volatility of returns. Moreover, the risk-free rate used to calculate the Sharpe ratio is 2.25%, which was the 1-year U.S. government bond rate at 10-11 July, 2008. The 1-year risk-free rate is chosen because of the 1-year horizon on the investments. The Sharpe ratios' sensitivity to the risk-free rate is examined in Sec. 4.2. The other portfolio statistics follow immediately from the realization of the simulations.

Examining performance using real data from July 11, 2008 to July 10, 2009 could be regarded as a study of ERC performance when including VIX Futures during a tail event, and is more of supplemental character. However, the reason for not looking at real data from the years following 2009 is that the estimation of the model parameters should be considerably skewed (outdated) when taking the financial crisis into account. The comparability would therefore be lost.

Chapter 4

Results

In this chapter, the results from the Matlab simulations are presented, with some details deferred to Appendix A and Appendix B. The chapter is divided into a first part showing a (partial) realization of the simulations, followed by performance statistics for portfolios with and without VIX Futures positions for different rebalancing frequencies. Finally, the roll over equal risk contribution portfolios' performance on historical data is presented.

4.1 Simulated Processes

Performing simulations using the code presented in Appendix D gives trajectories for the evolution of the S&P 500 Index, the VIX Index, VIX Futures maturing in three months and 10-year zero coupon bond prices over one year starting from a hypothetical July 11, 2008. As stated in Ch. 3, 1000 simulations are used for each rebalancing frequency. In Figure 4.1, five realizations of the simulations are shown to get the reader an overview of the realized processes used to allocate capital with the equal risk contribution approach.

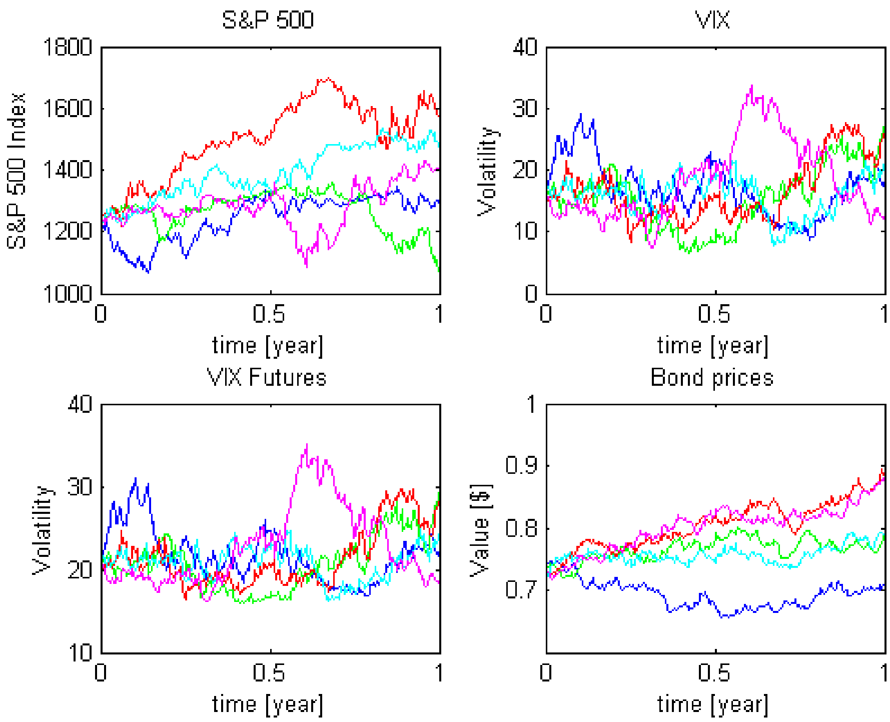


Figure 4.1: Graphs showing an example of 5 simulated S&P 500 processes, VIX processes, VIX Futures processes and bond price processes. Parameters are as stated in Ch. 3.

4.2 Predetermined Rebalancing Frequency

In this section, the performance statistics of the simulated portfolios with and without VIX Futures positions are presented for different roll over rebalancing frequencies. Simulations are performed in accordance with the approach presented in Ch. 3. Table 4.1 shows the performance statistics for portfolios including allocation to VIX Futures, while Table 4.2 is without VIX Futures positions. The asset allocation weights are presented in Table B.1 and Table B.2 in Appendix B.

Table 4.1: One year performance parameters for equal risk weights unlevered portfolio including VIX Futures, predetermined rebalancing frequency. Based on yearly return of 1000 simulations.

Rebalancing Frequency	Daily	Weekly	Monthly	Quarterly	Semi-annually	Annually
Avg yearly return	7.82%	7.08%	6.70%	6.63%	6.13%	5.50%
Annualized vol of ret	7.67%	7.62%	7.82%	8.02%	8.26%	8.42%
Sharpe ratio	0.73	0.63	0.57	0.55	0.47	0.39
SE(Sharpe ratio)	0.036	0.035	0.034	0.034	0.033	0.033
Worst drawdown	-15.47%	-14.59%	-14.45%	-18.06%	-20.66%	-33.20%
Highest return	31.32%	30.18%	30.73%	31.92%	30.81%	33.69%
Skewness	0.06	0.10	0.09	-0.11	-0.14	-0.16
Kurtosis	3.16	3.06	2.95	2.94	2.96	3.64

Table 4.2: One year performance parameters for equal risk weights unlevered portfolio without VIX Futures, predetermined rebalancing frequency. Based on yearly return of 1000 simulations.

Rebalancing Frequency	Daily	Weekly	Monthly	Quarterly	Semi-annually	Annually
Avg yearly return	4.85%	4.91%	4.92%	5.06%	5.10%	4.99%
Annualized vol of ret	8.03%	8.00%	8.13%	8.49%	8.72%	9.15%
Sharpe ratio	0.32	0.33	0.33	0.33	0.33	0.30
SE(Sharpe ratio)	0.032	0.033	0.033	0.033	0.033	0.032
Worst drawdown	-21.08%	-20.38%	-17.59%	-21.33%	-19.46%	-40.16%
Highest return	31.95%	31.07%	28.48%	28.46%	30.17%	32.00%
Skewness	-0.02	0.01	0.02	-0.13	-0.17	-0.28
Kurtosis	3.02	2.98	2.77	2.80	2.89	3.73

A comparison between the performance of the portfolios with and without VIX Futures positions shows that an inclusion of VIX Futures seems credible. For all rebalancing frequencies considered, the ERC portfolio including VIX Futures has more preferable performance statistics. In particular, the average yearly return is higher and the annualized volatility of return is lower consistently when including VIX Futures contracts. In turn, this results in a higher Sharpe ratio. Moreover, the worst drawdown is less negative when using VIX Futures apart from rebalancing at a semi-annual basis and the highest yearly returns are fairly equal between the two portfolios. In addition, for a normal distribution, the skewness is zero and the kurtosis equals three. By construction of the simulated asset classes using Wiener processes this is expected to hold for the returns, and is verified in Table 4.1 and Table 4.2.

Using the normal properties of the asymptotic distribution of the Sharpe ratio gives a possibility to perform a hypothesis testing of the null hypothesis H_0: "the Sharpe ratio with VIX Futures is less than or equal to the Sharpe ratio without VIX Futures" against the alternative hypothesis H_1: "the Sharpe ratio with VIX Futures position is higher than without VIX Futures positions". Forming the difference between the Sharpe ratios with VIX Futures SR_{VIX} and without VIX Futures SR_{WO} gives again a normal distributed variable. The variance of this variable has an upper bound given by $SE(SR_{VIX})^2 + SE(SR_{WO})^2$, since the covariance between the Sharpe ratios is positive. Under H_0, this yields the test variable z_{obs} for the null hypothesis H_0: $SR_{VIX} - SR_{W0} \leq 0$ against H_1: $SR_{VIX} - SR_{W0} > 0$ given by:

$$z_{obs} = \frac{\widehat{SR}_{VIX} - \widehat{SR}_{W0} - 0}{\sqrt{SE(\widehat{SR}_{VIX})^2 + SE(\widehat{SR}_{WO})^2}} \tag{4.1}$$

The values of the test statistics and the corresponding p-values calculated from the fact that the asymptotic distribution of the test variable is standard normal is given by Table 4.3

Table 4.3: Values of test variables z_{obs} and corresponding p-values for test of the null hypothesis H_0: "the Share ratio with VIX Futures is less than or equal to the Sharpe ratio without VIX Futures" against H_1: "the Sharpe ratio with VIX Futures positions is higher than without VIX Futures positions".

Rebalancing Frequency	Daily	Weekly	Monthly	Quarterly	Semi-annually	Annually
z_{obs}	8.51	6.23	5.07	4.64	3.00	1.96
p-value	$<10^{-10}$	$2{\cdot}10^{-10}$	$2{\cdot}10^{-7}$	$2{\cdot}10^{-6}$	$1{\cdot}10^{-3}$	0.025

As seen in Table 4.3, the null hypothesis is rejected in favour of the alternative hypothesis. I.e. the Sharpe ratio of the portfolio including VIX Futures position is significantly higher than the Sharpe ratio of the portfolio without VIX Futures contracts.

4.2.1 Transaction Costs Estimation

In this section, an estimate of the effect of including transaction costs when rebalancing the portfolios is examined. In Table 4.4 and Table 4.5, the total yearly rebalancing turnover is presented for the portfolios containing VIX Futures contracts and the portfolios without VIX Futures contracts respectively.

Table 4.4: One year average reallocation as a multiple of initial capital for equal risk weights unlevered portfolio including VIX Futures, predetermined rebalancing frequency. Standard deviation within parenthesis. Based on yearly return of 1000 simulations.

Rebalancing Frequency	Daily	Weekly	Monthly	Quarterly	Semi-annually	Annually
S&P 500	10.19	4.67	1.44	0.44	0.17	-
	(1.07)	(0.71)	(0.40)	(0.21)	(0.11)	-
Government bonds	10.28	4.81	1.49	0.46	0.17	-
	(1.00)	(0.71)	(0.41)	(0.21)	(0.11)	-
VIX Futures	1.14	0.47	0.13	0.04	0.02	-
	(0.16)	(0.08)	(0.02)	(0.02)	(0.01)	-

Table 4.5: One year average reallocation as a multiple of initial capital for equal risk weights unlevered portfolio without VIX Futures, predetermined rebalancing frequency. Standard deviation within parenthesis. Based on yearly return of 1000 simulations.

Rebalancing Frequency	Daily	Weekly	Monthly	Quarterly	Semi-annually	Annually
S&P 500	8.65	4.08	1.30	0.42	0.17	-
	(0.87)	(0.62)	(0.36)	(0.20)	(0.11)	-
Government bonds	8.65	4.08	1.30	0.42	0.17	-
	(0.87)	(0.62)	(0.36)	(0.20)	(0.11)	-

Using the approach in Sec. 3.2.1, in particular the assumption that the transaction cost is a fraction of the total transaction amount (0.05 percent) and that it is the same for all assets, the aggregated transaction cost is the sum of the turnovers of all assets multiplied by the transaction cost brokerage. This yields the following effect on returns and Sharpe ratios:

Table 4.6: Estimated effect on average yearly return and Sharpe ratio of portfolios with and without VIX Futures position when including transaction costs.

Rebalancing Frequency	Daily	Weekly	Monthly	Quarterly	Semi-annually	Annually
Avg yearly ret w VIX	-1.08%	-0.49%	-0.15%	-0.05%	-0.02%	-
Sharpe ratio w VIX	-0.14	-0.06	-0.02	-0.01	-0.00	-
Avg yearly ret no VIX	-0.87%	-0.41%	-0.13%	-0.04%	-0.02%	-
Sharpe ratio no VIX	-0.11	-0.05	-0.02	-0.01	-0.00	-

As seen in Table 4.6, the inclusion of transaction cost has a slightly larger impact on the portfolios with VIX Futures positions than the portfolios without VIX Futures.

4.2.2 Robustness of Results

The robustness of the portfolio performance is based on the stability of the simulation outcomes. In this report this regards returns, standard deviations of returns and the risk-free rate. Thus, ideally all parameters in the simulations should be tweaked to obtain robust results. However, the assumption of not updating the parameters gradually might be of higher importance. If this method is used in reality, the parameters may be updated perhaps every night or every week. This would mitigate the need for tweaking, and therefore a study of changes of the input parameters is disregarded. Nevertheless, one idea of how this could be done is by adding white noise terms to the parameters with variance based on the standard errors of the parameter estimation.

The parameter that could be changed without tweaking the simulations is the risk-free rate used in the calculation of the Sharpe ratio. As could be seen in Figure 4.2, the portfolio including VIX Futures has a higher Sharpe ratio than the portfolio without VIX Futures positions for a wide range of risk-free rates. Since the lines are almost parallel, they are practically equally sensitive to changes in the risk-free rate. The graph only shows the sensitivity based on the 1-year daily rebalance performance, but the same analogy holds for all rebalance frequencies.

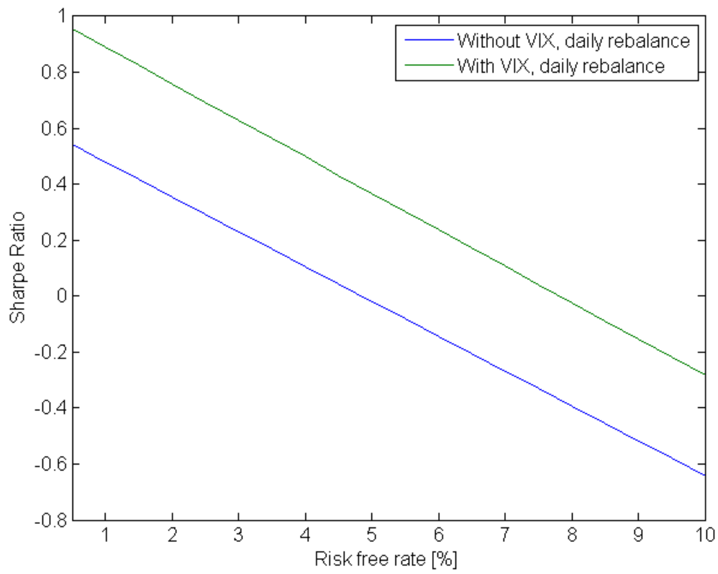


Figure 4.2: Graph showing how the Sharpe ratio of the different portfolios with daily rebalancing depend on the risk-free rate.

4.3 Test of Performance on Historical Data

Lastly, an equal risk contribution capital allocation approach applied to the actual performance of the different asset classes starting from July 11, 2008 and one year forward is considered. The results are shown in Table 4.7. The reason why only yearly returns are stated is because one time series can be considered as a single data point. As seen in the table, the yearly return is higher when including VIX Futures for all rebalancing frequencies considered. The changes between positive and negative returns are due to the timing of the allocation of capital.

Table 4.7: One year performance parameters for equal risk weights unlevered portfolio, predetermined rebalance frequency. Based on actual daily data during the period 11 July, 2008 - 10 July, 2009.

Rebalancing Frequency	Daily	Weekly	Monthly	Quarterly	Semi-annually	Annually
Ret with VIX Fut	8.75%	-0.88%	8.46%	4.09%	-0.10%	-1.88%
Ret without VIX Fut	0.58%	-2.33%	0.26%	-0.62%	-4.73%	-5.05%

Chapter 5

Discussion

The findings and results presented in Ch. 4 is analyzed in this chapter. Firstly, a thorough examination and interpretation of the results is performed. Thereafter, implications and conclusions of this study are reported. Finally, some suggestions for further studies are presented.

5.1 Interpretation of Results

The results in Ch. 4 demonstrate the benefits of including VIX Futures positions in the portfolios considered. The main result is the relatively higher annual returns and lower annualized volatilities for the VIX portfolio, yielding higher Sharpe ratios. The difference in Sharpe ratios holds for a wide range of risk-free rates as shown in Figure 4.2. Moreover, by forming the differences between the Sharpe ratios and testing the null hypothesis H_0: "the Sharpe ratio with VIX Futures is less than or equal to the Sharpe ratio without VIX Futures" against H_1: "the Sharpe ratio with VIX Futures position is higher than without VIX Futures positions", the VIX portfolio is shown to have a higher Sharpe ratio with p-values ranging from 10^{-10} to 10^{-7} for daily to monthly rebalances and 0.025 at yearly rebalancing frequencies. Therefore, the conclusion that including VIX Futures yields a higher Sharpe ratio could be said to be statistically significant.

In Sec. 4.2.1, transaction costs were examined to investigate if the previous conclusions would turn ambiguous under less restricted assumptions. Naturally, the calculations show that the returns and Sharpe ratios are lowered when a transaction cost is introduced. The important part, though, is that they also show that the relative difference in transaction cost is so small that it does not affect the interpretation that including VIX Futures increases the portfolio performance. Hence the conclusion drawn in the previous paragraph remains reliable.

A further note is the impact of the simplifications made in regards to margin accounts. As mentioned, the negligence of interest rate results in a somewhat lower return for the portfolio including VIX Futures. If an investor could receive an interest rate of, say, 2 % and the average allocation to VIX Futures (see Appendix B) is roughly 2 %, then the impact is a lowered return of 0.04 % annually. Including this effect would slightly increase the Sharpe ratio of the VIX portfolio.

The results on historical data again show that the portfolio including VIX Futures clearly outperforms the portfolio without VIX Futures for all rebalancing frequencies. As year 2008-2009 can be regarded as a tail event, the results show the hedging benefits from including VIX in a portfolio. This can be seen by comparing the average return of the VIX portfolio with the average return of the portfolio without VIX, for both simulations and historical data. Across rebalancing frequencies, the average return of the VIX portfolio on historical data was 5.06 % higher than the portfolio without, compared to the corresponding figure of 1.67 % for the simulations. Hence, this suggests that the effect of including VIX Futures is more beneficial when extreme events such as the crisis in 2008-2009 occur. Note however that the returns for weekly rebalance is considerably lower than for monthly and daily, which breaks a pattern of returns decreasing with rebalancing frequency. This is due to unlucky market timing, since the weekly rebalances contains all the rebalancing points used in monthly rebalancing, and a subset of the 20 % of the points used in daily rebalancing.

A noteworthy result is how the Sharpe ratio depends on reallocation frequency when VIX Futures is added. As can be seen in Ch. 4, the Sharpe ratio increases with rebalancing frequency for the VIX portfolio but is relatively constant for the portfolio consisting of only S&P 500 and government bonds. There is still an increase in Sharpe ratio with rebalancing frequency for the VIX portfolio when transaction costs are added. This shows the importance of an actively managed portfolio when VIX is included in the ERC approach.

If an investor holds a well-diversified portfolio (which could be assumed to hold approximately in real life examples), the diversification benefit of adding another asset class should be zero. Since the considered portfolios contains two and three asset classes respectively, an important discussion is to try to estimate to what extent the benefits are purely attributable to diversification. Logically, the VIX Index and thereby VIX Futures should not have a positive expected return, in fact it should be costly to hold. Therefore, it should be safe to say that the expected return of VIX Futures is *at least* not larger than the expected return of a bond investment, or the expected return of S&P 500. Since the same amount is invested in both portfolios, no extra expected return should be gained from including the VIX Futures. Hence, the benefits from diversification should be in the form of (at best) the same return, but with lower volatility. For diversification purposes, the asset added should ideally be negatively correlated with the portfolio, which it is. An upper limit of the diversification benefit is thus the total decrease in volatility, roughly 0.40 %. This will have an impact on the Sharpe ratios of at most 0.03, which is not enough to alter the interpretation that including VIX Futures have a significant positive effect on portfolio performance.

5.2 Implications and Conclusions

The economic interpretation and conclusion of the results is that: Including VIX Futures as an asset class in an equal risk contribution portfolio consisting of S&P 500 and government bonds substantially enhances the portfolio performance. Based on yearly returns, this study shows that the Sharpe ratio when including VIX Futures increases within a range of 30 % to 128% (corresponding to a range between 0.09 and 0.40) for rebalancing frequencies spanning from annually to daily, respectively. The conclusion holds significantly when effects from transaction costs and diversification benefits are accounted for.

The conclusion that VIX Futures increases the portfolio performance is also shown to be true when historical data of the corresponding period is examined. The study on historical data suggests that inclusion of VIX Futures has hedging benefits when tail events occur.

The practical implication is therefore that investors and asset managers using the equal risk contribution approach could benefit from including VIX Futures as an asset class in their portfolio.

5.3 Suggestions for Further Studies

An extension of this analysis is to build models and include other types of asset classes that could be relevant for investors using an equal risk contribution approach. Another highly relevant extension is to examine how often the parameters are needed to be re-estimated to better capture the movements of the market, and then run new simulations with these updated parameters.

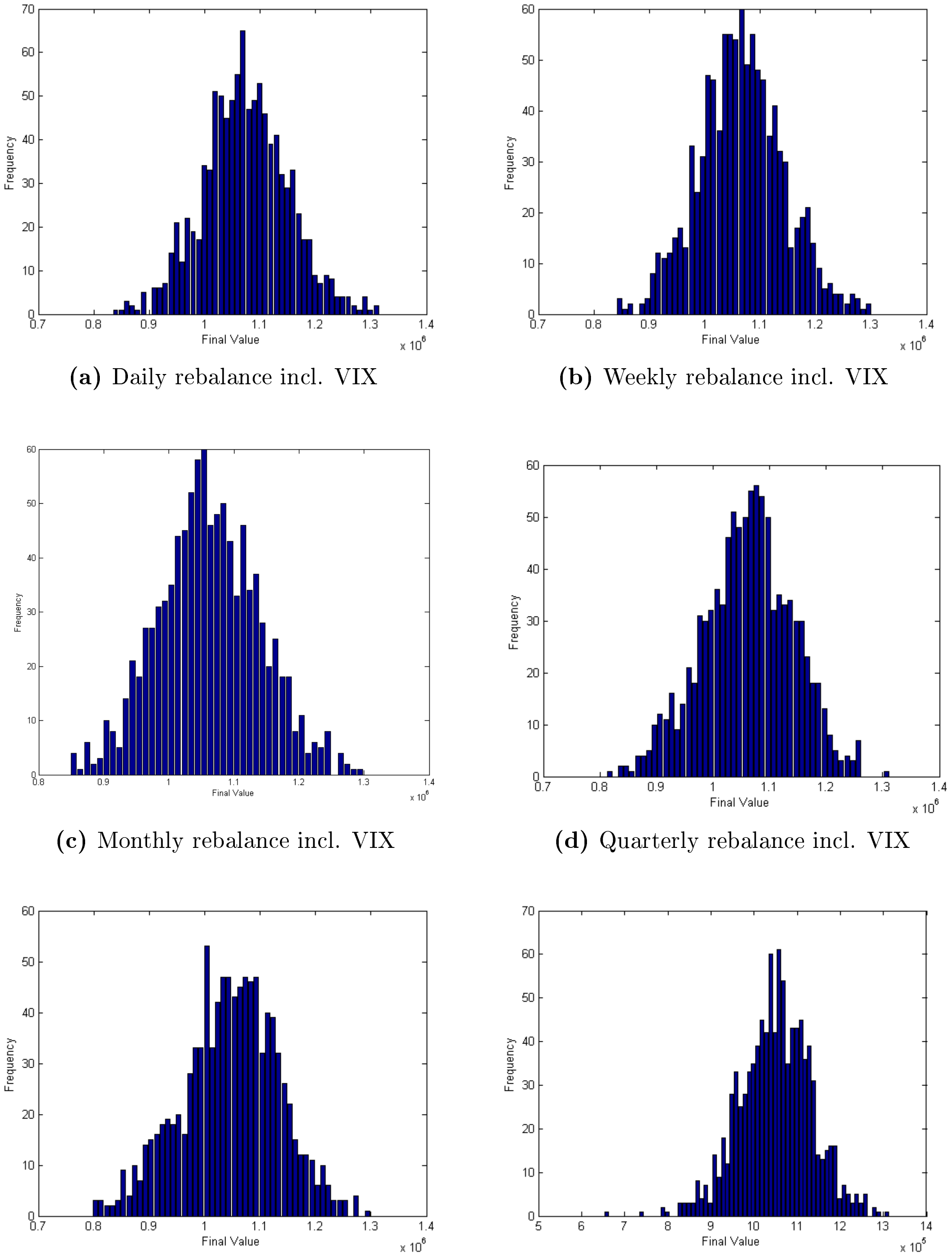
A further extension is to investigate some of the new views on ERC, e.g. the "Forecast Uncertainty Hedge" rule, and examine what impact VIX Futures could have on these approaches. One first step would be to simply add VIX Futures and see how the portfolio performance changes. The forecast uncertainty approach also requires development of a new rebalancing rule depending on the assessed uncertainty in the investments considered. Thus, a second step could be to investigate what role the VIX Index and VIX Futures could have in a reallocation or forecast rule, as it is an instrument indicating the implied market volatility in the near future.

Appendix A

Results - Graphs

For reference, histograms over average annual returns for different rebalancing frequencies are presented below. Each histogram is based on 1000 data points, and performance statistics on every simulation can be found in Chapter 4.2.

For each specific rebalancing frequency both a portfolio including VIX Futures and without VIX Futures was simulated simultaneously. This means that they have the same underlying 1000 processes of S&P 500, government bonds and VIX Futures.



(a) Daily rebalance incl. VIX

(b) Weekly rebalance incl. VIX

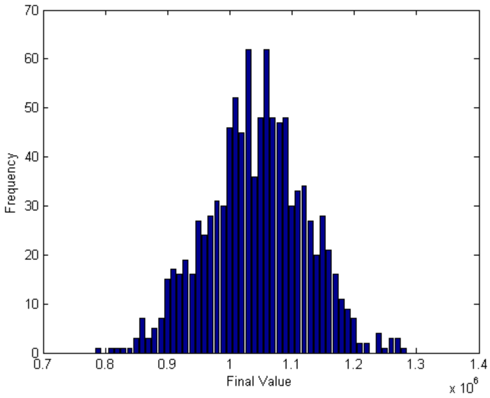
(c) Monthly rebalance incl. VIX

(d) Quarterly rebalance incl. VIX

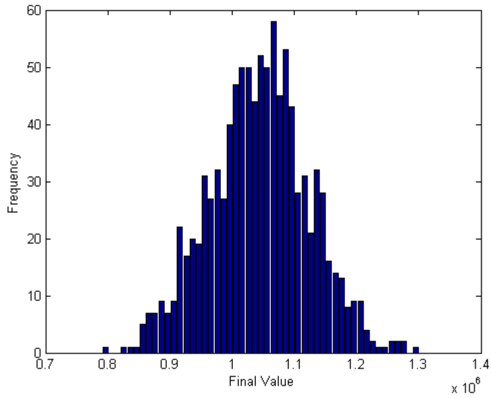
(e) Semi-annual rebalance incl. VIX

(f) Annual rebalance incl. VIX

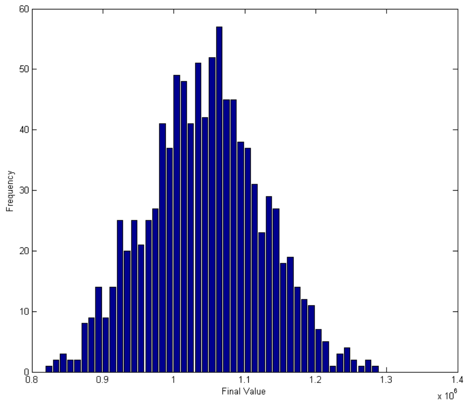
Figure A.1: Histogram over returns for S&P 500, government bond and VIX Futures portfolio using 1000 simulations, initial value $ 1 million.



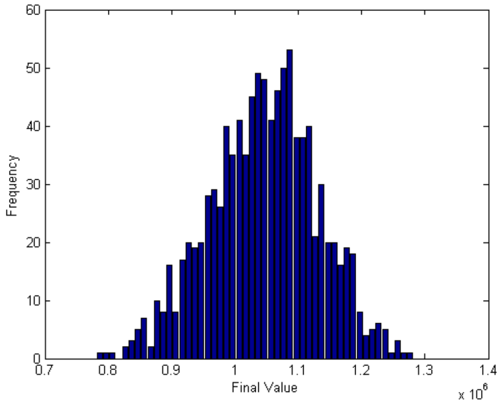
(a) Daily rebalance without VIX



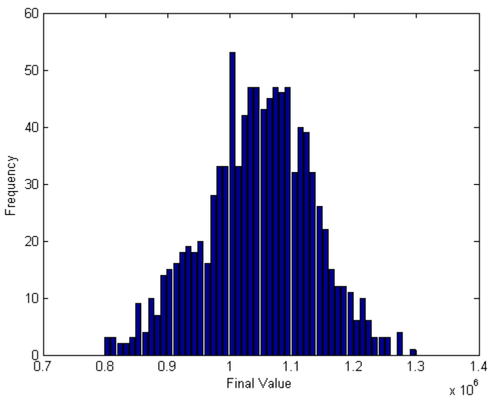
(b) Weekly rebalance without VIX



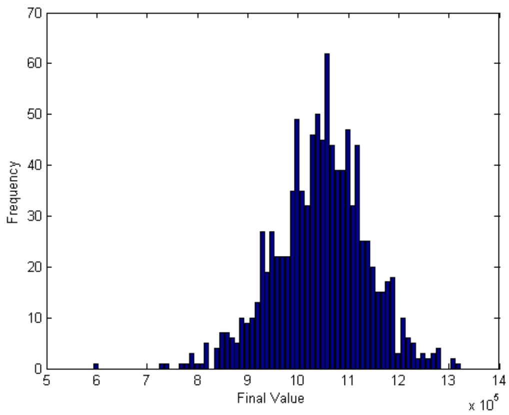
(c) Monthly rebalance without VIX



(d) Quarterly rebalance without VIX



(e) Semi-annual rebalance without VIX



(f) Annual rebalance without VIX

Figure A.2: Histogram over returns for S&P and government bond portfolio using 1000 simulations, initial value $ 1 million.

Appendix B

Results - Tables

In this part, weight statistics of the different assets for both portfolios with and without allocation to VIX Futures are shown. In Table B.1, the minimum, average and maximum allocation of the average position taken during each investment year is presented for S&P 500, VIX Futures and 10 year government bonds. Likewise, Table B.2 regards the portfolios when investing in S&P 500 and government bonds only. Moreover, Table B.3 and Table B.4 show the corresponding weights when historical data is used instead of simulated data.

Table B.1: Yearly weight data for equal risk weights unlevered portfolio including VIX Futures, predetermined rebalancing frequency. Based on yearly return of 1000 simulations.

Rebalancing Frequency	Daily	Weekly	Monthly	Quarterly	Semi-annually	Annually
Min w S&P	18.12%	19.32%	19.49%	16.53%	13.77%	8.06%
Avg w S&P	34.55%	34.60%	34.92%	35.60%	36.59%	37.46%
Max w S&P	54.19%	55.33%	55.82%	64.11%	67.14%	72.71%
Min w VIX Fut	1.83%	1.79%	1.52%	-0.81%	0.54%	0.28%
Avg w VIX Fut	2.54%	2.50%	2.32%	1.96%	1.55%	1.05%
Max w VIX Fut	3.57%	3.64%	3.65%	4.41%	3.69%	3.30%
Min w Bonds	42.02%	41.82%	40.90%	33.75%	31.48%	26.20%
Avg w Bonds	62.91%	62.90%	62.77%	62.44%	61.86%	61.49%
Max w Bonds	79.31%	78.35%	78.70%	81.46%	84.66%	90.64%

Table B.2: Yearly weight data for equal risk weights unlevered portfolio without VIX Futures, predetermined rebalancing frequency. Based on yearly return of 1000 simulations.

Rebalancing Frequency	Daily	Weekly	Monthly	Quarterly	Semi-annually	Annually
Min w S&P	19.39%	20.09%	20.40%	18.93%	16.51%	12.72%
Avg w S&P	36.42%	36.48%	36.85%	37.44%	38.42%	39.87%
Max w S&P	58.96%	58.47%	58.24%	63.51%	70.22%	71.72%
Min w Bonds	41.04%	41.53%	41.76%	36.49%	29.78%	28.28%
Avg w Bonds	63.58%	63.52%	63.15%	62.56%	61.58%	60.13%
Max w Bonds	80.61%	79.91%	79.60%	81.07%	83.49%	87.28%

Table B.3: Daily weight data for equal risk weights unlevered portfolio including VIX Futures, predetermined rebalancing frequency. Based on actual daily data during the period 11 July, 2008 - 10 July, 2009.

Rebalancing Frequency	Daily	Weekly	Monthly	Quarterly	Semi-annually	Annually
Min w S&P	7.01%	10.17%	11.35%	11.35%	23.96%	-
Avg w S&P	27.57%	27.54%	27.67%	26.65%	31.58%	35.39%
Max w S&P	53.85%	49.67%	38.12%	36.31%	36.31%	-
Min w VIX Fut	1.04%	1.15%	1.43%	1.43%	1.43%	-
Avg w VIX Fut	3.57%	3.58%	3.59%	2.29%	2.19%	2.58%
Max w VIX Fut	8.40%	7.53%	5.70%	2.61%	2.61%	-
Min w Bonds	41.31%	45.60%	57.29%	61.08%	61.08%	-
Avg w Bonds	68.87%	68.88%	68.73%	71.05%	66.23%	62.03%
Max w Bonds	91.62%	88.03%	86.06%	86.05%	74.61%	-

Table B.4: Daily weight data for equal risk weights unlevered portfolio without VIX Futures, predetermined rebalancing frequency. Based on actual daily data during the period 11 July, 2008 - 10 July, 2009.

Rebalancing Frequency	Daily	Weekly	Monthly	Quarterly	Semi-annually	Annually
Min w S&P	9.85%	9.85%	11.32%	11.32%	23.75%	-
Avg w S&P	28.18%	28.38%	29.12%	25.42%	30.04%	33.18%
Max w S&P	58.97%	57.86%	40.08%	33.78%	33.78%	-
Min w Bonds	41.03%	42.14%	59.92%	66.22%	66.22%	-
Avg w Bonds	71.82%	71.62%	70.88%	74.58%	69.96%	66.82%
Max w Bonds	90.15%	90.50%	88.68%	88.68%	76.25%	-

Appendix C

Mathematics

In this part, more details of the fundamental mathematics are presented. Moreover, a technical note about the calculation of the VIX Index is given.

C.1 Short on Arbitrage Theory

In financial mathematics; probability theory in general and martingale theory in particular is crucial for the understanding of how to determine arbitrage-free prices of financial derivatives (contingent claims). Using martingale theory, e.g. as in Björk (2009), one can show that:

In order to avoid arbitrage, the contingent claim X maturing at time T must be priced at time t according to the formula:

$$\Pi(t; X) = S_0(t) E^Q \left[\frac{X}{S_0(T)} \middle| \mathcal{F}_t \right] \tag{C.1}$$

where Q is a martingale measure for $[S_0, S_1, .., S_N]$, with S_0 as the numeraire.

Especially, one can choose the money market account (the bank account) $B(t)$ or some stock traded in the market, say $S_1(t)$, as numeraire. Informally, the numeraire is a way to quote the prices. $\mathcal{F}_t$ is called a *filtration*, and can basically be thought of as the information at hand at time t. I.e. the time t price of the derivative with payoff X at time T is dependent on the market information known at time t.

Another crucial part of the pricing of contingent claims is probability measures. In practice one observes the real world, which in probability theory is called the subjective probability measure P. However, in pricing formula (C.1) with $S_0(t) = B(t)$ one calculates under the probability measure Q, which is referred to as the *risk-neutral* martingale measure. This is because Q is formed by removing *the market price of risk* from the subjective probability measure P. Informally, Q assigns zero probability to the same events as P (is *equivalent*) and has the characteristic that "one prefer more to less" (there is no free lunch with vanishing risk).

C.2 VIX Index Calculation

As Stated by the CBOE [5], the calculation of the VIX Index is based on prices of options written on S&P 500 stocks. The formula used is given by:

$$\mathrm{VIX} = 100\sqrt{\frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{RT} Q(K_i) - \frac{1}{T}\left[\frac{F}{K_0} - 1\right]^2} \tag{C.2}$$

where T is the time to maturity, F the forward index level derived from index option prices, K_0 the first strike price below F, K_i the strike price of the i:th out-of-the-money option, ΔK_i the interval between strike prices, R the risk-free interest rate and $Q(K_i)$ the midpoint of the bid-ask spread for each option with strike K_i. For a further description of how to calculate the variables consult the white paper by CBOE [5].

Appendix D

MATLAB Code

In this chapter, the MATLAB code used in the simulations in the report is presented.

D.1 Main Program - parameters.m

This program runs the simulations and calculates the rebalances.

```
clear,clc
warning off;

% Parameters from Zhu & Lian SV—model
global theta kappa sigmav etav rho
theta=0.0444;
kappa=2.2680;
sigmav=0.3856;
etav=—2.016;
rho=—0.7533;

%Parameters for short rate model
global ahat sigmar
ahat=0.02;
sigmar=0.01;
```

```
% Time parameters
T=1;
n=252; % Trading days on a year

% Initial conditions
SP0=1239.49;%July 11,2008
VIX0=0.22;% A reasonable starting point for the ...
   considered time period
vol0=0.0225; % -"-

% Stock parameters
gamma=0.037; %Gramham and Harvey
r0=0.0126;

% Yield curve July 11, 2008
maturity=[0 1/12 1/4 .5 1 2 3 5 7 10 20 30];
yield = [r0 0.0138 0.0162 0.0202 0.023 0.0259 0.0288 ...
   0.0327 0.0355 0.0396 ...
            0.0457 0.0453];

finalValuesVIX = [];
finalValuesWO = [];
reallocVIX = zeros(3,nrsims);
reallocWO = zeros(2,nrsims);
wavgVIX = [];
wavgWO = [];
for simulationnr=1:nrsims
    simulationnr
    % Simulate S&P 500, VIX-Futures prices and bond prices
    [t,SP,SPr,VIX,Futures,Bonds]= ...
    simulateSPaVIX(SP0,VIX0,gamma,r0,vol0,maturity,yield,T,n);
```

```
% Allocations
wVIX = [];
wWO = [];

% Risk Parity allocation
Futret = zeros(1,n);
Bondret = zeros(1,n);
for i=2:length(t)
    Futret(i-1) = ...
       (Futures(i)-Futures(i-1))*100*1000/5500;
    %100 for right price, 1000 multiplier, 5500 ...
       initial margin
    Bondret(i-1) = (Bonds(i)-Bonds(i-1))/Bonds(i-1);
end

% Length of time interval for estimation of ...
   covariance matrix
estParam = 7;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%          With VIX          %%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% First week is for calibration
estimates = [SPr(1:estParam); Futret(1:estParam); ...
   Bondret(1:estParam)]';
global estSigma
estSigma = cov(estimates);

% Finding the weight for the risk parity portfolio
w0=[0.4 0.2 0.7];
w = fmincon(@optFUN,w0,[],[],[],[],[],[],@constraints);
```

```
w = w/sum(w);

wVIX = [wVIX w'];

% Initial Capital
V = 1000000;

% Stock allocation
nrStocks=V*w(1)/SP(estParam);

% Futures specificatoions
initialMargin = 5500;
nrFutures = V*w(2)/initialMargin;
oldFuturesPrice = Futures(estParam);

% Bond allocation = V(3)
nrBonds = V*w(3)/Bonds(estParam);

% Rolling-over rebalancing
rebalancefq = 1;
realloclopVIX = zeros(3,1);

for j=estParam+1:rebalancefq:length(t)-1
    estimates = [SPr(j-estParam:j); ...
       Futret(j-estParam:j); ...
        Bondret(j-estParam:j)]';
    global estSigma
    estSigma = cov(estimates);

    % Time j value
    V = nrStocks*SP(j)+((Futures(j) ...
        -*100*nrFutures*1000+w(2)*V)+nrBonds*Bonds(j);
    % *1000 for Futures contract since $10/tick
```

```
        oldFuturesPrice = Futures(j);

        % Finding the weight for the risk parity portfolio
        w0=w;
        w = fmincon(@optFUN,w0,[],[],[],[],[],[],@constraints);


        w = w/sum(w);



        % Measuring average reallocation
        realloclooopVIX = realloclooopVIX + abs(w0-w)';


        wVIX = [wVIX w'];

        % Rebalancing
        nrStocks = w(1)*V/SP(j);
        nrFutures = w(2)*V/initialMargin;
        nrBonds = w(3)*V/Bonds(j);
    end

    V = nrStocks*SP(length(t))+((Futures(length(t))- ...
        oldFuturesPrice)*100*nrFutures*1000+w(2)*V)+...
        nrBonds*Bonds(length(t));

    % Final Value without transaction costs
    finalValuesVIX = [finalValuesVIX V];
    wavgVIX = [wavgVIX [mean(wVIX(1,:)) ...
       mean(wVIX(2,:)) mean(wVIX(3,:))]'];
    reallocVIX(:,simulationnr) = realloclooopVIX;
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%          Without VIX       %%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% First week is for calibration
estimates = [SPr(1:estParam); Bondret(1:estParam)]';
global estSigma
estSigma = cov(estimates);

% Finding the weight for the risk parity portfolio
w0=[0.4 0.7];
w = fmincon(@optFUN,w0,[],[],[],[],[],[],@constraintsWO);


w = w/sum(w);

realloclooopWO = zeros(2,1);
wWO = [wWO w'];

% Initial Capital
V = 1000000; % Initial capital

% Stock allocation
nrStocks=V*w(1)/SP(estParam);

% Bond allocation
nrBonds = V*w(2)/Bonds(estParam);

for j=estParam+1:rebalancefq:length(t)-1
    estimates = [SPr(j-estParam:j); ...
       Bondret(j-estParam:j)]';
    global estSigma
    estSigma = cov(estimates);
```

```
        % Time j value (no leverage)
        V = nrStocks*SP(j)+nrBonds*Bonds(j);

        % Finding the weight for the risk parity portfolio
        w0=w;
        w = fmincon(@optFUN,w0,[],[],[],[],[],[],@constraintsWO);


        w = w/sum(w);

        realloclooopWO = realloclooopWO + abs(w0-w)';
        wWO = [wWO w'];

        % Rebalancing
        nrStocks = w(1)*V/SP(j);
        nrBonds = w(2)*V/Bonds(j);
    end

    V = nrStocks*SP(length(t))+nrBonds*Bonds(length(t));

    % Final Value without transaction costs
    finalValuesWO = [finalValuesWO V];
    wavgWO = [wavgWO [mean(wWO(1,:)) mean(wWO(2,:))]'];
    reallocWO(:,simulationnr) = realloclooopWO;
end

% Plotting histrograms over the final values
figure(1);
edges=min(min(finalValuesVIX),min(finalValuesWO)):...
0.01*e+06:max(max(finalValuesVIX),max(finalValuesWO));
ans = histc(finalValuesVIX, edges);
bar(edges, ans)
xlabel('Final Value')
```

```
ylabel('Frequency')

figure(2);
edges=min(min(finalValuesVIX),min(finalValuesWO)):...
    0.01*e+06:max(max(finalValuesVIX),max(finalValuesWO));
ans = histc(finalValuesWO, edges);
bar(edges, ans)
xlabel('Final Value')
ylabel('Frequency')

% Calculating Sharpe Ratio
rf = 0.0225;
(mean(finalValuesVIX/1000000-1)-rf)/std(finalValuesVIX/1000000-1)
(mean(finalValuesWO/1000000-1)-rf)/std(finalValuesWO/1000000-1)
```

D.2 Simulations - simulateSPaVIX.m

This program is called to simulate the stochastic processes of VIX Futures, S&P 500 and 10 year Government bonds.

```
function[t,SP,SPr,VIX,Futures,bonds] = ...
    simulateSPaVIX(SP0,VIX0,gamma,r0,Vol0,yieldmaturity,yield,T,n)
global theta kappa sigmav etav rho ahat sigmar
global a b kappaQ thetaQ tau

%Parameters
kappaQ=etav+kappa;
thetaQ=kappa*theta/kappaQ;
tau=30/365;
a=(1-exp(-kappaQ*tau))/(kappaQ*tau);
b=thetaQ*(1-a);
```

```
% Time parameters
dt = T/n;
t = dt * [0:n];

% Setting up variables
lSPr = zeros(1,n+1); %S&P returns
SP = zeros(1,n+1); %S&P value
SP(1)=SP0;
lSPr(1) = 0;

VIX = zeros(1,n+1);
VIX(1) = VIX0;

Vol = zeros(1,n+1);
Vol(1) = Vol0;

r = zeros(1,n+1);
r(1) = r0;

% Generate Wiener increments, first row S&P and second ...
   row volatility V
dW = zeros(2,n+1);
covar=[1 rho; rho 1];
for i = 1:n+1;
   dW(1:2,i) = mvnrnd(zeros(2,1),covar)*sqrt(dt);
end

% Generate Wiener increments for short rate
dWr = zeros(2,n+1);
for i = 1:n+1;
   dWr(i) = randn*sqrt(dt);
end
```

```
% Calculate forward rates curve using linear ...
   interpolation, using 10 years
% for long-term government bonds
simMat = 10; % 10 years maturity time
simMatN = simMat/dt;
obsbonds = zeros(1,simMatN);
time = [0:dt:simMat];

% Calculate forward rates curve using linear interpolation
fwdrate = zeros(1,simMatN+1);
for i = 1:simMatN+1;
   fwdrate(i) = interp1(yieldmaturity,yield,time(i));
end

% Approximate the derivative of the forward rate ...
   curve, (only 1 year needed)
fwdrateT = zeros(2,n+1);
for i = 2:n;
   fwdrateT(i) = (fwdrate(i+1)-fwdrate(i-1))/(2*dt);
end

% Calculation of log(S&P-returns) and VIX
for i = 2:n+1;
  Theta=fwdrateT(i-1)+sigmar^2*B(0,t(i-1))...
      *exp(ahat*t(i-1))+ahat*(fwdrate(i-1)+sigmar^2...
      *(B(0,t(i-1)))^2/2);
  r(i)=r(i-1)+(Theta - ahat*r(i-1))*dt+sigmar*dWr(i-1);
  Vol(i)=abs(Vol(i-1)+kappa*(theta-Vol(i-1))*dt+...
      sigmav*sqrt(Vol(i-1))*dW(2,i-1));
  lSPr(i)=(r(i-1)+gamma)*dt+sqrt(Vol(i-1))*dW(1,i-1);
  VIX(i)=sqrt(a*Vol(i)+b);
end
```

```
% Converting S&P returns to values
SPr=exp(lSPr);
SP(1)= SP0;
for i = 2:n+1
     SP(i)=SP(i-1)*SPr(i);
end
SPr=SPr-1;

% Calculation of observed long-term government bond ...
   prices, maturing in 10
% years
for i = 1:simMatN
    obsbonds(i)=exp(-trapz(time(i:length(time)),...
    fwdrate(i:length(fwdrate))));
end
obsbonds(length(obsbonds))=1;

% Calculation of simulated bond prices
bonds = zeros(1,n+1);
for i=1:length(t)
   bonds(i) = ...
      (obsbonds(1)/obsbonds(simMatN-i))*exp(B(t(i),simMat)...
   *fwdrate(i)-(sigmar^2/(4*ahat))*(B(t(i),simMat))^2...
   *(1-exp(-2*ahat*t(i)))-B(t(i),simMat)*r(i));
end

% Calculation of Futures prices F(t,T,VIXt)
Futures=[];
for i=1:length(t)
    timeToM = 3/12;
    Futures = [Futures FuturesPrice(timeToM,VIX(i))'];
```

```
%times 100 to get VIX—futures prices as quoted at CBOE
end
```

D.3 Function Files

In this section, short files are collected which are used to alleviate the coding of the other programs.

D.3.1 Futuresprices.m

This program calculates the Futures price of a VIX contract.

```
function out=FuturesPrice(timeToMIn,VIX0in);
global timeToM VIXt

% Parameters
VIXt=VIX0in;

% Calculation of Futures Price
out=[];
for i=1:length(timeToMIn)
    timeToM=timeToMIn(i);
    out = [out 1/(2*sqrt(pi))*quadgk(@fun,0,inf)];

end
```

D.3.2 Fun.m

This program calculates the integrand of the Futures price integral.

```
function output = fun(x)
global theta kappa sigmav
global a b kappaQ
global timeToM VIXt

% Calculation of Integrand of Futures Price
C=(-2*kappa*theta/(sigmav^2))*log(1+sigmav^2*(-x*a)...
*(exp(-kappaQ*timeToM)-1)/(2*kappaQ));
D=2*kappaQ*(-x*a)./(sigmav^2...
*(-x*a)+(2*kappaQ-sigmav^2*(-x*a))*exp(kappaQ*timeToM));
f=exp(C+D*(VIXt^2-b)/a);
output=(1-exp(-x*b).*f)./(x.^1.5);
```

D.3.3 optFUN.m

This is the function that is minimized when allocation capital according to an equal risk contribution approach.

```
function out=optFUN(win)
global estSigma
out = sqrt(win*estSigma*win');
```

D.3.4 constraints.m

This is the constraints used when optimizing the equal risk contribution portfolio including VIX Futures.

```
function [out, eq]=constraints(win)
out=[-(log(win(1))+log(win(2))+log(win(3)));
    -win(1); -win(2); -win(3)];
eq=[];
```

D.3.5 constraintsWO.m

This is the constraints used when optimizing the equal risk contribution portfolio not including VIX Futures.

```
function [out, eq]=constraintsWO(win)
out=[-(log(win(1))+log(win(2)));
    -win(1); -win(2)];
eq=[];
```

D.3.6 b.m

This function is called when calculating the 10 year government bond prices.

```
function output = B(t,T)
global ahat
output = (1/ahat)*(1-exp(-ahat*(T-t)));
```

References

[1] T. Björk, 2009, *Arbitrage Theory in Continuous Time*, 3rd edition, Oxford Univ. Press, Oxford.

[2] K. Black, 2006, *Improving Hedge Fund Risk Exposures by Hedging Equity Market Volatility, or How the VIX Ate My Kurtosis*, Journal of Trading, Vol 1, No.2, 6-15

[3] Board of Governors of the Federal Reserve System, *Historical Data, 10-year U.S. Government Securities*, available at `http://www.federalreserve.gov/releases/h15/data.htm`, accessed April 13, 2013.

[4] Chicago Board Option Exchange, *CBOE Volatility Index (VIX) Futures*, available at `http://cfe.cboe.com/Products/Spec_VIX.aspx`, accessed April 14, 2013.

[5] Chicago Board Option Exchange, *the CBOE Volatility Index - VIX*, available at `http://www.cboe.com/micro/vix/vixwhite.pdf`, accessed April 14, 2013.

[6] Chicago Board Options Exchange, *VIX Historical Price Data*, available at `http://www.cboe.com/micro/vix/historical.aspx`, accessed April 15, 2013.

[7] Chicago Board Options Exchange, *VX - CBOE S&P 500 Volatility Index (VIX) Futures Price and Volume Detail*, available at `http://cfe.cboe.com/Data/HistoricalData.aspx`, accessed April 13, 2013.

[8] Y. Choueifaty, T. Froidure and J. Reynier, 2013, *Properties of the Most Diversified Portfolio*, Journal of Investment Strategies, Vol 2, No.2, 49-70

[9] J. Duan and C. Yeh, 2007, *Jump and Volatility Risk Premiums Implied by VIX*, Journal of Economic Dynamics and Control, Vol 23, 2232-2244

[10] Federal Reserve Bank of St.Louis, *S&P500 Stock Price Index (SP500)*, available at `http://research.stlouisfed.org/fred2/series/SP500/downloaddata`, accessed April 13, 2013.

[11] J.R. Graham and C.R. Harvey, 2008, *Equity Risk Premium in 2008: Evidence from the Global CFO Outlook Survey*, Corporate Finance: Valuation, Capital Budgeting & Investment Policy eJournal, Vol. 07

[12] L.A. Grzelak, C.W. Oosterlee and S. van Weeren, 1990, *Extension of Stochastic Volatility Equity Models with Hull-White Interest Rate Process*, Quantitative, Vol 12, No.1, 89-105

[13] J. Hull and A. White, 1990, *Pricing Interest-Rate-Derivative Securities*, Review of Financial Studies, Vol 3, 573-592

[14] W. Lee, 2010, *Risk-Based Asset Allocation: A New Answer to An Old Question?*, Journal of Portfolio Management, Vol 37, No. 4, 11-28

[15] A. Lo, 2002, *The Statistics of Sharpe Ratios*, Financial Analysts Journal, Vol 58, No. 4, 36-52

[16] S. Maillard, T. Roncalli and J. Teiletche, 2009, *On the Properties of Equally-Weighted Risk Contributions Portfolios*, available at `http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1271972`, accessed April 3, 2013.

[17] H. Markovitz, 1952, *Portfolio Selection*, Journal of Portfolio Management, Vol 7, No.1, 77-91

[18] P. Rappoport and N. Nottebohm, J.P. Morgan Asset Management, 2012, *Improving on Risk Parity*, available at `http://www.jpmorganinstitutional.com/directdoc/jpmorgan/am/ia/documents/improving_on_risk_full_report`, accessed April 3, 2013.

[19] Y. Romahi and K.S. Santiago, J.P. Morgan Asset Management, 2012, *Diversification – still the only free lunch?*, available at `http://am.jpmorgan.co.uk/institutional/_documents/jpmam-25021-diversification-still-the-only-free-lunch.pdf`, accessed April 3, 2013.

[20] M. Schulmerich, 2010, *Real Option Valuation: The Importance of Interest Rate Modelling in Theory and Practice*, 2nd edition, Springer Verlag, New York

[21] W. Sharpe, 1966, *Mutual Fund Performance*, The Journal of Business, Vol 39, No.1, 119-138

[22] U.S. Department of the Treasury, *Daily Treasury Yield Curve Rates*, available at `http://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yield`, accessed April 3, 2013.

[23] S.P. Zhu and G.H. Lian, 2012, *An Analytical Formula for VIX Futures and Its Applications*, the Journal of Futures Markets, Vol. 32, No.2, 166-190