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An Assessment of the BGM-model Swap Option Pricing Performance in the Swedish Interest Rate Market*

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ABSTRACT

In this thesis the ability of a full-factor and a two-factor BGM-model to determine current and predict future plain-vanilla swaption prices issued on the Stockholm Interbank Offered Rate (STIBOR) is assessed. The study is conducted on daily data from January 4 to December 30, 2005. The assessment is based on a simultaneous calibration of the BGM-model to market implied cap and swaption volatilities. The calibration is made with a parametric volatility structure and an instantaneous correlation calibration within both a full-factor and a two-factor framework. The calibrated model is used to determine current and predict future plain-vanilla swaption prices using Rebonato's approximative formula for swaption prices. The BGM-model is found to accurately recover prices of plain-vanilla swaptions in-sample and out-of-sample on the date of calibration. The full-factor model is found to slightly outperform the two-factor BGM-model in both accurately pricing current date swaptions and predicting future swaption prices. For both model specifications, the average pricing error is found to be in an acceptable range, both for the pricing of current date swaptions and the prediction of future date swaption prices. The ability of the BGM-model to predict future plain-vanilla swaption prices is also found to decay as the prediction horizon is increased.

Keywords: Interest Rate Options; BGM-model; STIBOR

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I. Introduction

Blessed is the one who has an accurate interest rate option pricing model. In the late 1980s Chemical Bank used an incorrect model to value interest rate caps which cost the bank approximately \$33 million. In 1997 the National Westminster Bank applied an inappropriate model to value swaptions; the total loss incurred was roughly \$130 million (Hull 2003). Based on market statistics, Longstaff, Santa-Clara, and Schwartz (2000) estimated that the total present value costs of following sub-optimal exercise strategies for American swaptions could be on the order of several billion dollars. Hence, it is crucial to have an accurate model when pricing interest rate derivatives.

Although initially developed for options on commodities, the Black-76 formula has since publication been used for pricing interest rate derivatives (Black 1976). It is the interest rate option equivalent to the Black and Scholes formula for equity options and it is even the market practice to quote swaption prices in implied Black-76 volatilities. Nevertheless, the academic community disliked the theoretical shortcomings of the Black-76 model. For a long period of time there simply was no interest-rate dynamics compatible with the Black-76 formula for caps and swaptions. Brace, Gatarek, and Musiela (1997) and Miltersen, Sandmann, and Sondermann (1997) provided a breakthrough when they presented a theoretical framework where the evolution of the market observable forward rates was directly modelled under the assumption that they were lognormally distributed. This made it possible to construct logically consistent and arbitrage free models with theoretical prices for caps, floors, and swaptions that were of the Black-76 form. The model is called the Libor Market Model or the BGM-model after the inventors. In the same year Jamshidian (1997) derived the BGM/J model, which models the swap rates by assuming that they follow log-normal processes. The BGM- and BGM/J-models both belong to the class of market models, i.e. interest rate models that directly model the movement of market-observable rates. The BGM-model allows traders to match the volatility and correlation functions of the model to market observed quantities such as the Black-76-implied cap and swaption volatilities. The direct link between model and real-world dynamics is believed to yield accurate pricing and hedging of interest rate contingent claims.

The thesis has a two-fold objective focused on the ability of the BGM-model to accurately price plain-vanilla swaptions. The first objective stems from the need to accurately price swaptions today that were not used in the BGM-model calibration procedure. The assessment of the pricing performance at the calibration date can be viewed as a way to measure the robustness of the model. In practice, pricing is important for market makers quoting illiquid swaptions which by market practice are priced relative to the liquid swaption prices that are used as input for model calibration. The second objective is founded on the fact that the development of the market models have mainly been driven by traders of complex derivatives (Rebonato 2003). These traders trade exotic interest rate derivatives, e.g. Bermudan swaptions. During the life of the trade, the position is hedged by trading in plain-vanilla options, such as e.g. swaptions. To minimize the transaction costs of the hedging trades, it is crucial that the model provides a realistic evolution of model-implied future prices of the plain vanilla swaptions. Hence, as the second objective of the thesis, the ability of the BGM-model to predict future plain-vanilla swaption prices is assessed.

Purpose

The purpose of the thesis is to assess the ability of a full-factor and a two-factor BGM-model to determine current and predict future plain-vanilla swaption prices issued on the Stockholm Interbank Offered Rate (STIBOR).

Contribution

The ability to price and predict future plain-vanilla swaption prices is crucial for traders of complex derivatives and market makers quoting swaption prices. Research efforts have been focused on pricing advanced interest rate options for a limited set of data. The research has often been restricted to pricing a particular interest rate option on a single day, see e.g. Alpsten (2003). This thesis is the first assessment of the BGM-model pricing performance for a time-series of cap and swaption data with the STIBOR as the underlying rate.

Delimitation and methods

The authors believe that a master's thesis should preferably be self-contained. Here, however, the reader who is not willing to take results on faith would have to command a level of stochastic calculus equivalent to that found in e.g. Karatzas and Shreve (1998). To make the thesis readable for a audience of students of business and administration, the details on the implementation of the BGM-model are deferred to the Appendices and the reader interested in the mathematical derivation of the BGM-model should consult the original papers by Brace, Gatarek, and Musiela (1997) and Miltersen, Sandmann, and Sondermann (1997).

Interest rate options have been found to exhibit implied volatilities that are dependent on the strike rate, i.e. they exhibit a volatility skew. Research has recently been directed towards incorporating the volatility skew into the BGM-model framework, see e.g. Andersen and Andreasen (2000) and Brigo and Mercurio (2003). The volatility skew is not taken into account in the BGM-model specifications used in the thesis.

Outline

The thesis is structured as follows. First, a short review of previous empirical research of the performance of the BGM-model with a particular focus on research conducted on instruments issued on the STIBOR is presented. Second, the main features of the interest rate market is described, the relevant terminology is introduced, and the most popular interest rate models including the BGM-model is presented to get a view of how the BGM-model relates to other interest rate models. Third, the tests that are used to assess the plain-vanilla swaption pricing performance of the BGM-model are specified. Fourth, the BGM-model is calibrated to market observed cap and swaption prices and the parameter stability of the calibrated BGM-model is examined. Fifth, the swaption pricing performance of the calibrated BGM-model is assessed. Further, the pricing performance across different swaption expiries and underlying swap tenors is assessed. Finally, the main results are summarized and directions for further research are given.

II. Previous Research

A relatively small amount of empirical research on the interest rate option pricing performance of the BGM-model has been conducted. This holds particularly true for interest rate options issued on the STIBOR. The research has instead been focused on theoretical issues rather than the actual empirical performance of the BGM-model. The lagging empirical research can in part be attributed to the difficulty of obtaining sufficient data to conduct meaningful studies. This is due to that most of the interest rate derivatives are traded in over-the-counter-markets, that often lack a mechanism for the systematic collection of data. Hence, the research has concentrated on advanced methods to price exotic derivatives for a relatively small amount of input data, e.g. for a set of Bermudan swaptions for a given day. Moreover, the incentives to examine the empirical performance of various interest rate models for a large amount of data are higher in the financial industry, where the economic implications of a superior pricing model could be substantial. To our knowledge, no study has been made on interest rate options issued on the STIBOR for a time-series of data.

A. Empirical BGM-model studies ex Sweden

De Jong, Driessen, and Pelsser (2001) examine the pricing performance of Libor and Swap Market Models using cross-sectional data on prices of US caplets and swaptions. It is shown that the Libor Market Model in general leads to better prediction of derivative prices than the Swap Market Model with the model specifications used in the study. Moreover, it is reported that models that are chosen to exactly match the prices of a set of contingent claims are overfitted. In contrast, more parsimonious models are reported to lead to better predictions of prices of derivatives that were not part of the calibration procedure. In Gupta and Subrahmanyam (2001) caps and floors are priced with one- and two-factor Hull-White models, one- and two factor Black-Karasinski models, five specifications of the Heath-Jarrow-Morton forward-rate model, and a one-factor BGM-model across different maturities and strike rates. The tests are conducted for a time-series of daily data from March to December 1998. A superior pricing performance for the one-factor lognormal models is reported, and two-factor models are reported to only marginally improve the pricing accuracy. Hull and White (1999) test the pricing performance of the BGM-model across a range of strike rates and conclude that the absolute pricing error of the BGM-model for caps is greater than for swaptions. The study is typical in scope with data limited to one day.

B. BGM-model studies based on the STIBOR

Jansson (1999) calibrates a three-factor BGM model and prices caps with analytical formulas; and European and Bermudan swaptions with Monte Carlo simulation. Market prices of caps and swaptions and historical data for the correlation between various segments of the swap curve is used in the calibration. Gustavsson (2001) examines the practical issues involved when pricing interest rate options with the BGM/J-model. Alpsten (2003) considers the pricing of Bermudan swap options with the BGM-model. The pricing performance of the BGM-model for Bermudan swaptions is reported to be feasible for selling options over-the-counter in the particular implementation, but not for real-time trading. The Monte Carlo simulation is based on Mersenne Twister pseudo-random numbers. Kaisajuntti (2003) calibrates the BGM-model to caps, floors, and swaptions and prices caps, swaptions, and Bermudan swaptions. Inspired by Winiarski (2003) a quasi Monte Carlo technique employing Sobol pseudo-random numbers is used. Compared to the Mersenne Twister sequence, the Sobol sequence is reported to substantially reduce the time needed for Monte Carlo simulations. Computationally tractable models that yield prices that are well within the typical bid/ask-spread in the markets are reported.

III. Theory

This section presents the necessary background on interest rate markets and models. The relevant instruments are defined and the similarities and differences between the instruments are established.

A. The interest rate swap market

An interest rate swap is an agreement to exchange a set of interest payments on a specific principal amount. The principal amount is never exchanged, i.e. it is just a notional principal amount. The interest rate swap market dates back to the early 1980s and has grown rapidly since then. The growth subsequently made swaptions one of the most important fixed income derivative products. Today, the cap and swaption markets are the two main interest rate option markets in the world. The average daily global turnover in OTC derivatives markets exceeded $1 trillion in 2004 (Bank for International Settlements 2005).

The swap market is structured in primary and secondary markets for swaps and swaptions and consists of a world-wide network of swap dealers. A network of brokers, who do not take positions for their own account, facilitate trading between dealers by providing a degree of anonymity for dealers who wish to trade with other dealers. The high volume traded makes the swap markets highly liquid.

An interest rate swap offers borrowers such as institutional investors the opportunity to exploit a perceived capital market imperfection or to swap the distribution of their assets and liabilities (Fabozzi 2000). The initial motivation for the interest swap-market was the potential credit-arbitrage possibilities available for borrowers. These arbitrages exist due to the differences between the quality spread between lower- and higher-rated credits in the fixed-rate market and the same spread in the floating-rate market. The argument for swaps is based on the principle of comparative advantage in international economics. Although a high-rated issuer has better opportunities in an absolute sense, i.e. could borrow at a better rate in both the fixed- and floating rate markets, it has a comparative advantage relative to a low-rated issuer in one market. As a part of the evolution of the market place, eventually so called swap options, or *swaptions*, on the underlying interest rate swaps were introduced

B. Terminology

Zero-coupon bonds are fundamental tools for describing the value of interest rate products. A zero coupon bond entitles the holder to a fixed cash flow at a future date T. Zero coupon bonds will be assumed to have a unit notional in the following. This implies that the fixed cash flow received at time T is a unit amount. By combining zero coupon bonds of different maturities, different types of financial products can be constructed. At time t the price of a zero-coupon bond is denoted by $P(t,T)$. The n-year *zero-rate*, short for zero-coupon rate, is the rate of interest earned on an investment that starts today and lasts for n years. The n-year zero rate is also referred to as the n-year *spot-rate*. A graph showing the zero rate as a function of maturity is known as the *zero curve*.

An interest rate is quoted together with a *daycount fraction*. The daycount fraction is defined as the number of days in the period divided by the number of days in the year. Different markets use different methods to calculate these figures. The daycount fraction will be denoted by τ in the following. The *London Inter Bank Offered Rate (LIBOR)* is the spot rate offered by banks for lending to other banks. It is the rate of interest that one London bank will offer to pay on a deposit by another bank. The Swedish equivalent of LIBOR is the *Stockholm Inter Bank Offered Rate (STIBOR)*. At the end of a time period of length τ the lender receives an interest payment equal to τL where τ is the daycount fraction and L is the STIBOR rate. For the 3-month STIBOR the actual/360 daycount convention is used.

The STIBOR is normally quoted as the rate for an n-month loan where n typically is three, six, nine,... months. The *discrete forward rate* is defined as

$$F_i(t) \equiv \frac{1}{\tau_i}\left(\frac{P(t,T_{i-1}) - P(t,T_i)}{P(t,T_i)}\right) \tag{1}$$

where τ_i is the daycount fraction for the period $[T_{i-1}, T_i]$ and the price of a zero-coupon-bond at time t with maturity T_i is $P(t, T_i)$. The time T_{i-1} is the maturity of the rate and the time period $T_i - T_{i-1}$ is called the *tenor* of the forward rate. The current time will be denoted $T_{-1} \equiv 0$ in the following. The forward rate structure is built up by the first spot rate up to time T_0 and M forward rates reset at $\{T_0, T_1, \ldots T_M\}$. Figure 1 depicts the relationship between spot and forward rates.

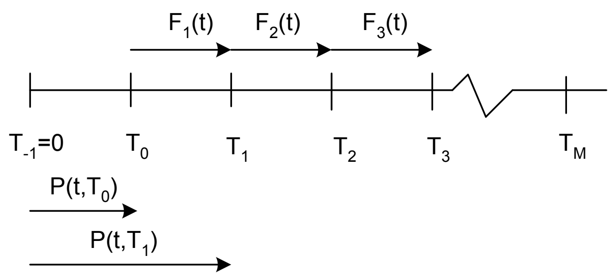

Figure 1. Spot and forward rates. Illustration of the relation between spot and forward rates.

The *instantaneous forward rate* at time t with maturity $T > t$ is denoted by $f(t, T)$ and is defined as $f(t,T) = -\frac{\partial \log P(t,T)}{\partial T}$. The instantaneous forward rate has no obvious equivalent in traded market instruments. The *instantaneous short rate* r, also called the *instantaneous spot rate* r, is the rate one earns on a riskless investment over an infinitesimally short period of time dt at time t. It is defined as $r(t) = f(t, t)$, i.e. it is a special case of the instantaneous forward rate. It is a mathematical abstraction and is not observable in the market.

B.1. Caps, floors, swaps, and swaptions

An *interest rate cap* is an interest rate option offered in the over-the-counter market. A cap offers the buyer an insurance against the floating-rate rising above a certain level. A *floor* offers the buyer an insurance against the floating-rate falling below a certain level. In practice, caps are used to hedge loans and floors are used to hedge deposits. Further, caps are used to hedge long-term interest rate risks. For a trader they present a limited risk, since only the premium can be lost. The thesis is only concerned with at-the-money (ATM) options, i.e. where the strike rate is equal to the current interest rate. The prices of ATM caps and floors, with equal maturities, are identical. The analysis is in the following hence restricted to cap prices only.

Caps are, by market practice, usually defined so that the initial rate does not lead to a payoff on the first reset date. A cap contract with strike K on a three-month interest rate is a portfolio of options called *caplets* on the quarterly interest payments. To be more precise, suppose an insurance that protects us from the forward LIBOR rates F_i rising above a level K is held. The payoff received at time T_i as a holder of a caplet is equal to

$$\tau_i \max\{F_i(T_i) - K, 0\}. \tag{2}$$

Since the payoff exactly matches the difference between the fixed payment $\tau_i K$ and the LIBOR payment $\tau_i F_i(T_i)$, this is a call option on the LIBOR rate. Since a cap is composed of caplets, it has a price that depends on both the level and the volatility of interest rates.

An *interest rate swap* is a contract where two parties agree to exchange a set of floating interest rate payments for a set of fixed interest rate payments. The fixed payments are referred to as the *fixed leg*. In this thesis, the floating rate payments are based on STIBOR rates and are referred to as the *floating leg*. The *swap rate* is the interest rate paid by the party responsible for the fixed payments. It is also called the *par swap rate* and it is the fixed rate at which the swap has a zero present value. The date T_α at which the swap starts is called the *expiry date*. The date T_β at which the last payment occurs is called the *maturity date*. The length $T_\beta - T_\alpha$ of the underlying interest rate swap is called the *tenor* of the swap. Figure 2 depicts an interest rate swap.

Swap options, or swaptions, are options on interest rate swaps. They give the holder the right to enter into a certain interest rate swap at a certain time in the future. In general, an option is called *European* if it only can be exercised

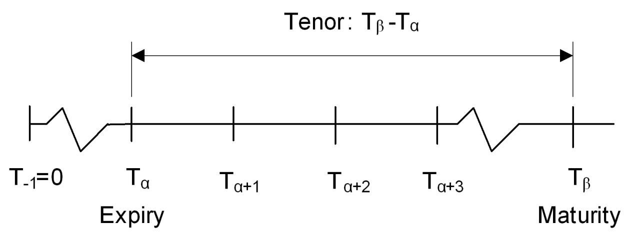

Figure 2. Interest rate swap. Illustration of the expiry- and maturity date and the tenor of an interest rate swap.

at maturity. The option is said to be *American* if it can be exercised at any time before expiry, and it is said to be *Bermudan* if it can be exercised at any one of a fixed set of dates. More precisely, an $T_\alpha \times (T_\beta - T_\alpha)$ European payer swaption with strike X is a contract which at the expiry date T_α gives the holder the right but not the obligation to enter into a swap of length $(T_\beta - T_\alpha)$ where he pays the fixed swap rate X, i.e. the strike rate, and receives the floating rate on a predetermined notional principal. Hence an European payer swaption is a call option on the floating interest rate. Since the notional principal multiplies through all equations, a notional principal of SEK 1 is assumed in the following. A receivers swaption is the right to to receive the fixed interest rate. A receiver swaption is hence a put option on the floating rate. In addition to the level and volatility of forward rates, the price of a swaption also depends on the correlation between forward rates.

C. Interest rate models

The main question answered by interest rate models is what the "fair" price of a certain interest rate derivative is. Ultimately, however, the market is the pricing engine. The Black-76 model has become the market standard for pricing interest options for reasons of simplicity. It yields an analytical formula for the market price of most plain-vanilla interest rate options. It is the market practice for pricing caps, floors, and swaptions and the market prices of the instruments are often quoted as Black-76 implied volatilities.

Different taxonomies can be used to classify interest rate models. First, *equilibrium models* are distinguished from *no-arbitrage models*. The main difference between these two types of models is that in the equilibrium model, the initial term structure is an output from the model, while in a no-arbitrage model it is an input to the model. Equilibrium models are linked to the state of the economy and the variables describing it. Some of these variables are not directly observable, such as investor preferences, and hence a equilibrium model does not automatically fit today's term structure. In contrast, no-arbitrage models are by design constructed to be exactly consistent with today's term structure without worrying about the economic rationale behind the model. Second, interest rate models can be specified as one-, two-, or even multi factor models, where the number of factors is the number of sources of randomness used in the model. This holds true both for equilibrium and no-arbitrage models.

Here, in addition to the Black-76 model, a taxonomy consisting of three different classes of models will be briefly introduced. Following Lee (2000) the similarities between models belonging to different classes in the chosen taxonomy will be reviewed. The three classes of models are the models for infinitesimal interest rates, the market models, and other types of models. The *models for infinitesimal rates* can be segmented into low dimensional models, such as short rate models that model the instantaneous short rate, and high dimensional models, such as the HJM-model (Heath, Jarrow, and Morton 1990). The high-dimensional models describe the dynamics of the entire yield curve using the instantaneous forward rates as the stochastic variable. The *market models* can be classified into LIBOR-market models, or equivalently BGM-models, and swap market models, or equivalently BGM/J-models. *Other types of models* include models such as the Markov functional models (Kennedy, Hunt, and Pelsser 2000). The thesis will not be concerned with exotic types of models and will thus not introduce models belonging to the third class any further.

A similarity shared by the low- and high dimensional infinitesimal rate models and the market models is that they all describe the evolution of the yield curve through a stochastic differential equation driven by a drift- and a diffusion term (Lee 2000). High-dimensional models for the infinitesimal rates and the market models share that they both model forward rates, with the difference that market models model the market observable, discrete forward rates. The following sections introduces the Black-76 model and the two other main classes of models, namely the models for infinitesimal rates and the market models.

C.1. The Black-76 model

The market practice for options on futures has been dominated by the model first presented in Black (1976). The Black-76 model is built on the same conceptual framework as the Black and Scholes stock option model developed in Black and Scholes (1973) and Merton (1973). The Black-Scholes analysis assumes constant, or deterministic, interest rates. Since the object under study here is an interest rate dependent derivative, the interest rate is the underlying variable instead of the stock. The assumption that the stock price follows a log-normal process is, however, replaced with the assumption that the forward rate follows such a process. The Black-76 model can be modified to price bond options, caps, and swaptions. The Black-76 formula is the market standard, and the prices of many products such as caps, floors, and swaptions are quoted as implied Black-76 volatilities. More precisely, given a market price for e.g. a swaption, the implied Black volatility is the volatility that when inserted in the Black-76 formula yields the market observed price.

In the Black-76 formula, each individual caplet is priced as a call on a log-normally distributed interest rate. The input parameters are the volatility of the interest rate, the strike price K, the time to the cashflow $T_i - t$ and two interest rates. The first interest rate is the underlying instrument and is defined as the current forward rate applying between times T_{i-1} and T_i. The second interest rate is used for discounting the caplet cash flow to present value and it is the yield on a zero-coupon bond maturing at time T_i. This yields a price of

$$
\begin{aligned}
\mathbf{Capl}_{\mathbf{i}}^{\mathbf{Black}}(t) &= \tau_i P(t,T_i)\{F_i(t)N[d_1] - KN[d_2]\},\ i=1,\dots,N \\
d_1 &= \tfrac{1}{\sigma_i\sqrt{T_i-t}}\left[\ln\left(\tfrac{F_i(t)}{K}\right) + \tfrac{1}{2}\sigma_i^2(T_i-t)\right] \\
d_2 &= d_1 - \sigma_i\sqrt{T_i-t}
\end{aligned} \tag{3}
$$

where the constant σ_i is known as the *Black volatility* for caplet i.

The Black-76 formula can also be applied to value European swaptions. The underlying in this case is the par swap rate $R_\alpha^\beta(t)$, which is the strike rate X for which the swap has a value of zero. It is assumed to follow a lognormal process. The accrual factor used for discounting is defined as $S_\alpha^\beta(t) = \sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)$. Using the accrual factor $S_\alpha^\beta(t)$ the forward swap rate can be written as

$$
R_\alpha^\beta(t) = \frac{P(t,T_\alpha) - P(t,T_\beta)}{S_\alpha^\beta(t)}, \quad 0 \le t \le T_\alpha. \tag{4}
$$

The Black-76 formula for a $T_\alpha \times (T_\beta - T_\alpha)$ payer swaption with strike X, swap rate $R_\alpha^\beta(t)$, and accrual factor $S_\alpha^\beta(t)$ is

$$
\begin{aligned}
\mathbf{PSN}_\alpha^\beta(t) &= S_\alpha^\beta(t)\{R_\alpha^\beta(t)N[d_1] - XN[d_2]\} \\
d_1 &= \tfrac{1}{\sigma_{\alpha,\beta}\sqrt{T_\alpha-t}}\left[\ln\left(\tfrac{R_\alpha^\beta(t)}{X}\right) + \tfrac{1}{2}\sigma_{\alpha,\beta}^2(T_\alpha-t)\right] \\
d_2 &= d_1 - \sigma_{\alpha,\beta}\sqrt{T_\alpha-t}
\end{aligned} \tag{5}
$$

Hence, the volatility of the par swap rate, the time to expiry, the time to maturity, and the accrual factor must be known to value the swaption as a typical call option payoff with the Black-76 formula.

C.2. Short- and infinitesimal forward rate models

In the short rate models, or equivalently spot rate models, the instantaneous short rate is modelled with a diffusion process. By integration, all other interest rates can be obtained. The main drawback of the short rate models is that the infinitesimal short rate cannot be observed in the real world. The pricing is further not consistent with the market practice of using the Black-76 formula in the sense that market rates often are hard to express in terms of short rates. As a consequence, it is hard to obtain a good fit of the model to observed cap and swaption prices holding information on the market price of risk. Finally, the model parameters are hard to interpret in economic terms. In the group of different short rate models the main difference between models is found in the form of the diffusion equation and the associated assumption on the distribution of the short rate. Further, short rate models can be equilibrium models or no-arbitrage models, depending on whether the current term structure is an output from or an input to the model. Table I depicts the diffusion equation, the assumption of the distribution of the short rate, and whether the model is of the equilibrium or no-arbitrage type.

The Vasicek (1977) model is among the first instantaneous short-rate models introduced and incorporates a mean-reverting process for the short rate. It has one major drawback; it provides a poor fit of the initial term structure of interest rates. There are simply not enough free parameters to provide a correct calibration to the prices of all bonds. In the case of an option on a bond, a slight mispricing in the bond price is accentuated in the option price which could be severely mispriced, due to its non-linear nature. To provide a better fit to the observed yield curve, Hull and White (1990) introduced a time-varying parameter in the Vasicek model. They extend the model by adding the no-arbitrage property and introducing two additional time-dependencies. Interest rates are assumed to be normally distributed, which has the drawback that they can become negative. The Hull-White model provides a better fit to both the currently-observed yield curve and the term structure of volatilities. The parameters a and σ are constants and θ is chosen to exactly fit the term structure of interest rates. The model can be further expanded by allowing for the parameters a and σ to be time-varying.

The Black-Derman-Toy-model is a no-arbitrage model that allows for incorporation information about both the current term structure of spot interest rates and the term structure of spot rate volatilities (Black, Derman, and Toy 1990). A lognormally distributed short rate is assumed, which ensures that the interest rate never becomes negative. The BDT-model is popular among practitioners, due

Table I
Short rate models

The table presents the diffusion processes for different popular short rate models.

Model	Diffusion equation	Distribution	Type
Vasicek	$dr = (b - ar)dt + \sigma dW$	Normal	Equilibrium
Hull-White	$dr = (\theta(t) - ar)dt + \sigma dW$	Normal	No-arbitrage
Black-Derman Toy	$dr = \theta(t)rdt + \sigma(t)rdW$	Lognormal	No-arbitrage
Black-Karasinski	$dr = (\theta(t) - a \ln r(t))rdt + \sigma(t)dW$	Lognormal	No-arbitrage
Cox-Ingersoll-Ross	$dr = (b - ar)dt + \sigma\sqrt{r}dW$	χ^2	Equilibrium

to its simplicity of calibration and because of the straightforward analytic results it offers. One drawback is that it has mutually dependent mean-reversion and volatility terms. In the no-arbitrage Black-Karasinski model it is also assumed that the interest rates are log-normally distributed (Black and Karasinski 1991). Normally, the model has constant parameters a and σ, but it can be expanded to allow for time-varying a- and σ-parameters. The BK-model is an extension to the BDT-model in the sense that it allows for more flexibility with independent parameters. Among market practitioners the prices obtained with the Hull-White and Black-Karasinski models are often compared as a robustness check, since they represent two different assumptions on the distribution of interest rates. The shared advantage of the BK and BDT models is that they provide a good fit to the prices actively traded in the markets. The shared disadvantage is that they have a non-stationary volatility structure, implying that the future term structure of volatility can be quite different from the term structure of volatility observed today.

Another model where the interest rate is sure to remain above or equal to zero, but that is an equilibrium model, is the CIR-model introduced in Cox, Ingersoll, and Ross (1985). It has the same mean-reverting drift as the Vasicek model, and assumes that the short rate follows a noncentral χ^2-distribution. Ho and Lee (1986) introduced an arbitrage free model with a normally distributed short rate with a time-varying drift. The prices of discount bonds of all maturities are taken as given and the evolution of the whole price vector is constructed to prevent arbitrage opportunities. The Ho-Lee model is both no-arbitrage and analytically tractable. It has, however, two main drawbacks in that it lacks mean reversion and that the volatility term structure is inflexible with all spot and forward rates having the same volatility.

Heath, Jarrow, and Morton (1990) derived an arbitrage-free framework for the evolution of the entire yield curve in continuous time. They used the instantaneous forward rates as building blocks and thus introduced the so called *forward rate models*. In the HJM-framework forward rates maturing at various fixed points in time are allowed to evolve simultaneously. Further, the HJM methodology is arbitrage free and thus matches the current term structure by construction. Hence, it does not lead to initially mis-priced underlying discount bonds as is the case with the equilibrium instantaneous short rate models. By appropriately choosing the volatility function of the forward interest rates the processes of the instantaneous short interest rate models can be treated as special cases of the HJM-model. For example, specifying the volatility as an exponential function of the time to maturity gives rise to the Ornstein Uhlenbeck process that is found in the Vasicek (1977) model. A drawback of the Heath-Jarrow-Morton framework is that often no analytically tractable formulas exist.

Note that when calibrating these models, the parameters must be inferred from market data. Traditional statistical techniques cannot be used, since the calculations involving the parameters are performed under the martingale measure, not the physical measure.

C.3. Market models

The first market model was developed by Brace, Gatarek, and Musiela (1997) and Miltersen, Sandmann, and Sondermann (1997). Jamshidian (1997) also contributed significantly with the first swap market model, the so called BGM/J-model. As often when several contributors independently develop a model, the model has several names; Europeans usually call it a LIBOR Market Model and the BGM/J-model is called a Swap Market Model. The development of the *market* *models* was the reaction to the difficulties encountered when calibrating one- and multidimensional infinitesimal models to market prices. Instead of instantaneous short or forward rates, the market models describe the discrete rates observable in the market. The BGM-model is a HJM type model with the additional requirement that the market quoted forward rates are log-normal for a particular tenor. Specifically, the BGM-model is developed in an arbitrage free framework so that the theoretical prices derived within the model have the structure of the Black-76 formula. This makes it possible to accurately match the prices observed in the market with the model parameters of the BGM-model. A sketch of the derivation of the BGM-model drift dynamics with a brief mathematical background can be found in the Appendix.

The main theoretical objection against the BGM- and BGM/J-models is that they are not compatible with each other. The assumptions made when the two models are derived are not consistent with each other. If forward rates are log-normally distributed, as assumed by the BGM-model, the forward swap rates cannot be lognormal. The latter is assumed by the BGM/J-model, which makes the models incompatible. The discrepancy, however, seems to be small. Rebonato (1999a) shows that the forward swap rates obtained from lognormal forward LIBOR rates are not far from being lognormal and it is the market practice to ignore the inconsistencies.

D. Users of and hedging with interest rate models

Term structure models are primarily used by relative-value bond traders, plain-vanilla option traders, and traders of complex derivatives (Rebonato 2003). The different users of term structure models, the object these users are concerned with, and the area of application is shown in Table II. Initially, the main users of interest rate models were plain-vanilla and relative-value bond traders. This has shifted towards institutions and funds that trade more complex interest rate derivatives. These currently drive the development of term structure models in general and market models such as the BGM-model in particular. An important area of use

Table II
Users of term structure models

User	Object	Area of use
Relative-value bond trader	Forward rates	Coupon-to-coupon arbitrage
Plain-vanilla trader	Forward rates	Delta hedging
Complex derivatives trader	Forward rate volatilities	Out-of-model vega hedging

of interest rate models is that of hedging. *In-model hedging* concerns hedging a derivative by taking positions in the underlying instrument to neutralize the price uncertainty caused by the stochastic movements of the underlying, i.e. obtaining delta neutrality $\Delta = \frac{\partial F}{\partial R} = 0$ where F is the price of the derivative and R is the underlying rate. *Out-of-model* hedging is the practice of taking positions to neutralize the sensitivity of a complex product to variations in input quantities that are assumed to be deterministic by the model. A trader of complex derivatives often strives to be left unaffected by changes in the volatility, that is defined as a deterministic function in several interest rate models. Hence, the trader aims to achieve vega neutrality, i.e. $\mathcal{V} = \frac{\partial F}{\partial \sigma} = 0$, by trading plain-vanilla options.

The market practice of out-of-model hedging is conceptually unsound, since it contradicts the assumption that the volatility is deterministic and known. In practice vega-hedging is common, model re-calibration is the standard, and the two concepts go hand in hand. If the market is assumed to be frictionless, the future prices of the hedging instruments at date $t > 0$ would be implied by the model calibration at time 0. If, in contrast, the future conditional prices of plain-vanilla options predicted at date 0 differs from the future prices actually observed, the model has to be re-calibrated. Thus the market practice of re-calibrating a model during the life of the complex trade is also flawed and closely related to the practice of vega-hedging. The difference between predicted and observed prices of hedging options leads to an extra cost that is not accounted for in the model. Hence, the use of a good model should lead to small differences between future market prices actually observed and the prices predicted by the model at the initial calibration date (Rebonato 2003). The objective of this thesis is to assess the magnitude of this difference for the BGM-model together with the ability to recover current date swaption prices.

IV. Methodology

The thesis has a two-fold objective focused on the ability of the BGM-model to accurately price plain-vanilla swaptions. The first objective is based on the need to accurately price swaptions today that were not used in the BGM-model calibration procedure. Pricing these swaptions will be referred to as out-of-sample pricing in the following. The pricing of swaptions used in the calibration is referred to as in-sample pricing. The second objective is to assess the ability of the BGM-model to predict future plain-vanilla swaption prices with the model calibrated to today's data.

First, the pricing performance at the current date is assessed by calibrating the BGM-model to caps and to a restricted set of swaptions. More precisely, the calibration does not use the full swaption matrix as input data. Instead, only swaptions from three columns in the swaption matrix are used. The columns in the matrix used for calibration corresponds to swaptions with an underlying swap tenor length of two, five, and nine years. The choice of columns is based on these columns representing the most actively traded swaptions. Further they span the largest available range of swap tenors for which data is available in the sample. This reduces the computational effort of the calibration procedure and is in line with market practice. In general, the swaption prices used as input should be chosen with respect to the type of the derivative that is priced. Finally, the swaptions that were not used as inputs in the model calibration are priced and the prices implied by the BGM-model are compared to the market observed swaption prices.

Second, to assess the ability of the BGM-model to predict the future prices of plain-vanilla swaptions conditional on market prices observed today, the framework developed in Gupta and Subrahmanyam (2001) is used. The predictive ability is important for traders of complex derivatives involved in the practice of re-balancing their portfolios to achieve vega-neutrality, i.e. neutralizing the position to changes in volatility. Moreover, the prediction ability has implications for assessing the market risk by calculating e.g. the Value-at-Risk for a portfolio of swaptions. To be in line with the market practice of frequently re-calibrating the model, two relatively short prediction horizons of one day and one week, are chosen. Given market data today, the swaption prices on the next day and after five days are predicted. The BGM-model is calibrated to cap and swaption prices and the term structure available today. The model parameters implied by the market are then used together with the term structure on the prediction day to calculate the swaption price predicted by the model. The accuracy of the model is assessed by comparing the predicted swaption price with the actual price observed in the market. Finally, the prediction horizon is extended in increments of one day up to a ten day prediction horizon, in increments of five days up to a 50 day prediction horizon, and in increments of 10 days to a 150 day prediction horizon to assess how the prediction ability changes when the prediction horizon is increased. To further assess the BGM-model performance and robustness, three model criteria are examined:

1. Parameter stability and model performance over time
2. Systematic biases in the pricing errors
3. The complexity and difficulty in estimating the model

An ideal model is expected to have parameters that are reasonably stable in time, to have no systematic biases in the pricing errors, and to be computationally tractable.

A. Model Calibration

The process of implying a model's parameters from the prices of traded instruments is called calibration. The specification of volatility and correlation functions determines the model-implied future prices of caplets and swaptions.

A.1. Historical versus implied correlation

Whether historical or market implied forward rate correlations should be used depends on which specification that is capable of producing accurate future prices of hedging instruments, i.e. plain-vanilla swaption prices (Rebonato 2003). The debate of which of the two choices is preferable is ongoing and Longstaff, Santa-Clara, and Schwartz (2000) demonstrated that the costs of suboptimal exercise strategies can be substantial. The trading community, predominantly trained in the Black-Scholes model framework, tends to prefer market implied correlations over historical estimates. However, that the implied method is superior is not self-evident. The validity of relying on market-implied quantities for calibration requires exact payoff replicability and hence market completeness. The three main arguments supporting the choice of relying on market implied data are the following.

1. The input functions to the model are deterministic for volatilities and correlations and are perfectly known by the market
2. The input functions to the model are deterministic for volatilities and correlations but are *not* perfectly known by the market, and hence additional option prices are necessary for perfect replication
3. Financial markets are informationally efficient, i.e. there is no systematic imbalance of supply and demand for plain-vanilla options

The first point of view is flawed, since deterministic volatilities and correlations invalidates the whole practice of vega-hedging. The second point of view relies on the ability to perfectly replicate the contingent claim. Perfect replication is difficult in practice, since the options necessary for replication often are illiquid and only available for a short range of maturities. Hence, perfect replication is not feasible, even if the volatilities and correlations are deterministic but not perfectly known from market observed plain-vanilla prices. The third point of view, with the underlying assumption that plain-vanilla market prices are informationally efficient, is not easily refuted. The issue of market efficiency is maybe the most debated issue in contemporary finance, and the state of the debate will not be addressed here. But it needs to be stressed that an implicit assumption of market efficiency is made when only market implied volatility and correlation specifications are used to calibrate the BGM-model (Rebonato 2003).

A.2. Calibration to implied volatilities and correlations

In this thesis the BGM-model is calibrated to implied market data. Since the BGM-model yields a cap pricing formula of the Black-76 type, the BGM-model volatility parameters can easily be determined by matching the model volatility functions to the market observed Black-76 cap volatilities.

The parametric form of the volatility functions is chosen in alignment with two desirable empirical properties. First, mean-reversion in the forward rate volatilities needs to be accounted for. Second, the evolution of the forward rate term structure should be time-homogeneous. Care needs to be taken to not overfit the data when a rich parametric structure is chosen (De Jong, Driessen, and Pelsser 2001). Since the price of a swaption, in contrast to the price of a cap, depends on the correlation between forward rates, a correlation structure needs to be defined. The additional parameters introduced in the correlation structure are determined by calibrating the BGM-model to swaption and cap prices. A simultaneous calibration to both cap and swaption prices is believed to yield a more parsimonious model, with a robust evolution of the volatility term structure. In general, a calibration with as few parameters as possible is desirable. There is a great deal of sense in the old saying that the more you fit, the less you explain. Yield curve models with a large number of factors, jumps, and stochastic volatility are all worthless if they are not or can not be properly calibrated. According to market practitioners, model calibration is often more important than model choice (Andreasen 2003). The parameterization chosen for the BGM instantaneous volatility is

$$\sigma_i^{Model}(t) = \eta_i h(t) = \eta_i \left((a(T_{i-1} - t) + d) e^{-b(T_{i-1}-t)} + c \right), \tag{6}$$

and the parameterization for the instantaneous correlation between forward rates F_i and F_j is $\rho_{i,j} = e^{-\beta|T_i - T_j|}$. Both a full-factor BGM-model and a two-factor model is examined, where Rebonato's formulation with correlation angles is used to perform the factor reduction from the full-factor to the two-factor model (Rebonato 1999b). Details on the two-factor model correlation calibration can be found in the Appendix. In the calibration, swaptions are priced using the Rebonato approximation of the swaption volatility. The details of the calibration procedure and an extended discussion on the choice of the volatility functions and the correlation structure can be found in the Appendix.

B. Empirical test of pricing accuracy

The objective of the empirical test is to assess the ability of the BGM-model to predict future swaption prices. This is achieved by the following procedure. The prices observed in the market of caps and swaptions at date t_i are used to calibrate the model and to recover the implied BGM-model parameters. These parameters and the term structure at date t_{i+k} are used to calculate the prices of swaptions predicted by the model at date t_{i+k}. In Figure 3 the methodology is depicted for $k = 1$.

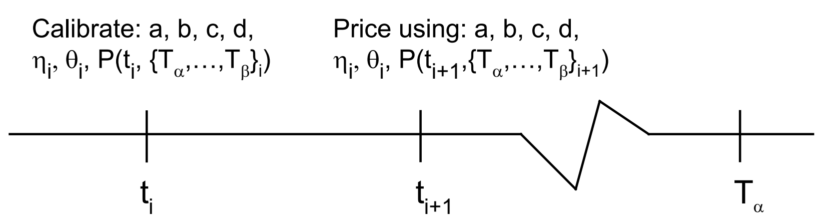

Figure 3. Methodology. The BGM-model parameters are implied from market prices of caps and swaptions and the term structure $P(t_i, \{T_\alpha, \ldots, T_\beta\}_i)$ at date t_i. These parameters $a, b, c, d, \eta_i, \theta_i$ are used together with the term structure $P(t_{i+1}, \{T_\alpha, \ldots, T_\beta\}_{i+1})$ at date t_{i+1} to assess the predicted model price at date t_{i+1}.

The observed market implied volatility at date t_{i+k} is then subtracted from the model-based implied volatility, and the absolute and relative pricing errors are computed. The procedure is repeated for each swaption and for each day in the sample, and

- the error $(IV_{t+k}^{Model} - IV_{t+k}^{Market})$,

- the absolute error ($|IV_{t+k}^{Model} - IV_{t+k}^{Market}|$),
- the percentage error ($IV_{t+k}^{Model}/IV_{t+k}^{Market} - 1$), and
- the absolute percentage error ($|IV_{t+k}^{Model}/IV_{t+k}^{Market} - 1|$)

are calculated and averaged over the sample. These error statistics are then further segmented by maturity and tenor to examine systematic biases and patterns in the pricing errors. In addition, possible systematic biases in the pricing performance are assessed by regressing the market price of the swaption on its model forecast price. The objective is to identify if the model is consistent with the data. More precisely, the regression

$$IV_t^{Market} = \beta_0 + \beta_1 IV_t^{Model} + \epsilon_t$$

is performed, where a good model should have a β_0-parameter insignificantly different from zero, a β_1-parameter insignificantly different from one, and a high R-square value. To avoid underestimating the standard deviations of the estimated parameters used in the hypotheses tests due to autocorrelation the Newey and West (1987) procedure is used. This procedure accounts for autocorrelation as well as heteroscedasticity. The regressions are performed for all data, and also for partitions of the data segmented on swaption tenors and maturities to examine systematic biases in the same manner as for the error statistics above. All of these statistics are evaluated for both a full-factor BGM model, and for a two-factor model obtained by factor reduction in order to evaluate the impact of the number of factors employed in the model.

Furthermore, for $k = 0$ the error statistics are calculated and the regressions are separately performed on the total data set, the data set used for calibration, and the data set containing only the swaptions that were not used for calibration. This to allow for the evaluation of the out-of-sample performance of the model on the day of calibration, which is important to assess whether the model is overfitted to the calibration data or not.

V. Data

The data consists of daily prices of Swedish krona caps and swaptions for 256 days for the period January 4 to December 30, 2005. The cap data is across seven maturities (1-, 2-, 3-, 4-, 5-, 7-, and 10-year) and the swaption data is across seven expiries (3-, 6-month, 1-, 2-, 3-, 4-, 5-year) and eight tenors (1-, 2-, 3-, 4-, 5-, 6-, 7-, 8-, 9-year). Certain combinations of option expiries and swap tenors were excluded due to stale data. The swaptions that were excluded due to stale data can be identified in Table IV where descriptive statistics for the swaption data is listed.

At each time instant the yield curve is defined as the zero-coupon interest rate as a function of time to maturity. Deposit rates with maturities of 1, 2, 3, 6, 9, and 12 months and swap rates with maturities of 2, 3, . . ., 10 years are used to construct the yield curve. The rates are first converted to zero-coupon rates. By interpolating between these zero rates the interest rate curve is obtained as a smooth function of maturity. For this study, zero rates maturing at each three month time interval up to ten years are interpolated. The choice of the interpolation method has been subject to much debate, but there is no consensus on a preferred method. Here, a method based on piecewise cubic spline interpolation is used.

The caplet volatilities are boot-strapped from actively traded cap prices. Since a cap can be seen as a portfolio of caplets, the price of a cap is the sum of the caplet prices constituting the cap. It is the market practice to quote cap prices as so called flat volatilities. This means that the market maker first decides how much each individual caplet is worth in terms of Black-76 volatility, i.e. each caplet is assigned its own volatility. These volatilities are then converted to prices with the Black-76 formula and summed up to obtain the price of the cap. Finally, the flat volatility is calculated as the volatility that yields the price of the cap if that volatility is assigned to all the caplets in the cap.

Since caplets on the STIBOR are reset on a quarterly basis, an assumption about the shape of the term structure of volatility has to be made and interpolation between the yearly data points for cap prices has to be carried out. A piecewise cubic spline interpolation is used since it preserves the humped shape of the term structure of volatility better than a linear interpolation. Moreover, an extrapolation is necessary for the first part of the term structure of volatility, since the one year cap price is not sufficient to back out the prices of the three caplets between $[0.25y; 0.5y]$, $[0.5y; 0.75y]$, and $[0.75y; 1y]$. After the cap prices have been obtained with a quarterly resolution by interpolation between yearly data points, the caplet volatilities are obtained by a bootstrap procedure.

A. Cap and caplet data interpolation

Interpolation is used to obtain cap prices with quarterly resolution from the cap prices with yearly resolution. The algorithm to bootstrap the caplet volatilities from the interpolated cap prices is as follows.

1. The price of the first caplet between $[0.25; 0.5]$ years is the price of the interpolated cap price with a 0.5 year maturity. The Black-76 formula yields the corresponding implied Black volatility.
2. The price of caplet number n between $[0.25n; 0.25 + 0.25n]$ years for $1 \leq n \leq 38$ is

$$\mathrm{Price}(\mathrm{Capl_n}) = \mathrm{Price}(\mathrm{Cap_n}) - \sum_{\mathrm{i=1}}^{\mathrm{n-1}} \mathrm{Price}(\mathrm{Capl_i})$$

3. The volatility σ_{Black} of the caplet number n with maturity T_n and strike K is solved for as the volatility implied by the price of the caplet according to

$$\mathrm{P}(\mathrm{Capl_n}) = \mathrm{Capl}^{\mathrm{Black}}(\mathrm{T_n}, \mathrm{F_n(t)}, \mathrm{K}, \sigma_{\mathrm{Black}}, \mathrm{P(t, T_n)}).$$

   Since all parameters except the volatility is known this corresponds to finding the solution of a one-variable equation. The Newton-Raphson method is used to find the root (Press, Teukolsky, Vetterling, and Flannery 1992).

B. Descriptive statistics

Tables III and IV present descriptive statistics for the cap and swaption data. The prices are expressed in implied Black volatilities. The average, median, minimum, and maximum price of the contracts are reported. The statistics indicate that the prices of caps and swaptions decrease, on average, with maturity. Further, there is a hump of the volatility term structure around the two year maturity.

Table III
Cap data descriptive statistics

The table presents the average, median, minimum, and maximum price in implied Black volatilities of the contracts over the sample period. The data consists of of cap prices across seven different maturities (1-, 2-, 3-, 4-, 5-, 7-, and 10-year). The sample consists of 256 trading days of daily data, from January 4 to December 30, 2005.

	Maturity (years)						
	1	2	3	4	5	7	10
Mean	19.1	23.0	23.0	22.4	21.7	20.7	19.6
Median	18.9	22.5	22.8	22.5	22.0	21.1	20.1
Min	13.0	20.2	19.8	19.0	18.3	17.3	16.4
Max	27.0	29.1	28.0	26.3	25.0	23.4	22.3

The main features of the data for a given day is described by presenting a snapshot of the data for a day chosen randomly in the sample, i.e. March 25, 2005. The volatility term structure on March 25, 2005 given by implied Black volatilities for caps with different maturities is depicted in Figure 4. The term structure for interest rates and zero coupon bonds (ZCB) is depicted in Figure 5. The surface of swaption volatilities across different maturities and swap lengths is depicted in Figure 6 below.

The dependence of volatilities on strike rate, not captured by the Black-76 model that assumes constant volatilities across different strike rates, is depicted in Figure 7 below. As mentioned in the delimitation section, the volatility skew is not accounted for in this thesis. However, since only at-the-money options are examined, the existence of the skew and the fact that the BGM model in its original form does not account for it is immaterial to this study.

The descriptive statistics for cap volatilities for the sample period January 4

Table IV

Swaption data descriptive statistics

The table presents mean, median, minimum, and maximum implied Black volatilities for ATM-swaptions with tenors between one and nine years and 3-, 6-month-, 1-, 2-, 3-, 4-, and 5-year expiries. The sample consists of 256 trading days of daily data, from January 4 to December 30, 2005. Some combinations of tenors and expiries are excluded due to stale data.

		Tenor (years)								
Expiry (years)		1	2	3	4	5	6	7	8	9
0.25	Mean	20.4	23.0	22.5	21.8	21.0	20.0	19.2	18.5	17.9
	Median	19.5	22.7	22.2	21.5	20.9	20.0	19.2	18.6	18.1
	Min	16.3	19.0	18.8	18.3	17.9	16.8	16.0	15.6	15.3
	Max	28.5	30.5	29.8	27.3	25.7	24.3	23.2	22.3	21.7
0.5	Mean	22.6	23.4	22.6	21.6	20.9	19.9	19.2	18.5	18.0
	Median	22.2	23.1	22.2	21.4	20.8	20.0	19.3	18.7	18.2
	Min	18.8	20.3	19.5	18.5	17.9	17.0	16.3	15.9	15.5
	Max	28.8	30.2	28.7	26.1	25.1	23.8	22.7	21.9	21.3
1	Mean	24.5	23.1	22.0	21.1	20.3		18.9	18.4	17.9
	Median	24.2	23.0	21.8	20.9	20.2		19.0	18.6	18.2
	Min	21.9	20.7	19.4	18.5	17.6		16.2	15.8	15.4
	Max	32.1	28.7	26.6	25.0	24.1		22.0	21.2	20.7
2	Mean			20.4	19.7	19.1		18.1	17.8	
	Median			20.5	19.7	19.3		18.5	18.1	
	Min			17.8	17.0	16.3		15.4	15.0	
	Max			23.2	22.3	21.8		20.6	20.1	
3	Mean	20.4	19.9	19.3	18.7		17.8	17.5		
	Median	20.4	19.9	19.3	18.8		18.0	17.7		
	Min	17.6	17.1	16.6	15.9		15.0	14.7		
	Max	22.8	22.6	22.1	21.6		20.5	20.1		
4	Mean	19.4	18.9	18.5	18.1	17.7	17.4			
	Median	19.5	18.8	18.3	18.1	17.8	17.6			
	Min	16.5	16.4	15.9	15.3	14.8	14.5			
	Max	22.1	21.7	21.3	20.7	20.4	20.3			
5	Mean	18.7	18.3	17.9		17.3				
	Median	18.4	18.1	17.7		17.6				
	Min	15.8	15.8	15.4		14.5				
	Max	21.5	21.0	20.7		20.6				

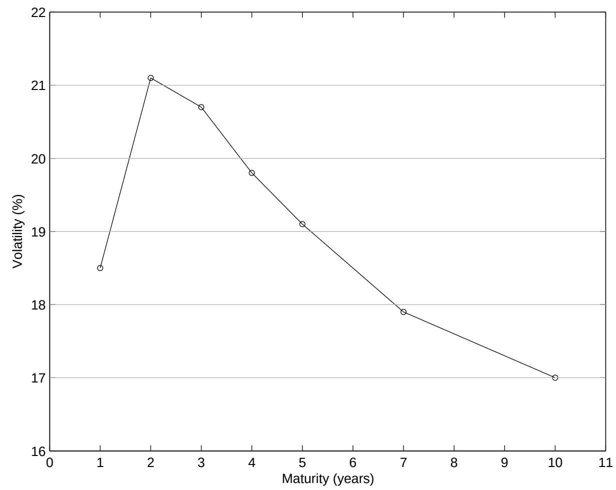

Figure 4. Volatility term structure. Implied Black cap volatilities for caps across 1-, 2-, 3-, 4-, 5-, 7-, and 10-year maturities on March 25, 2005.

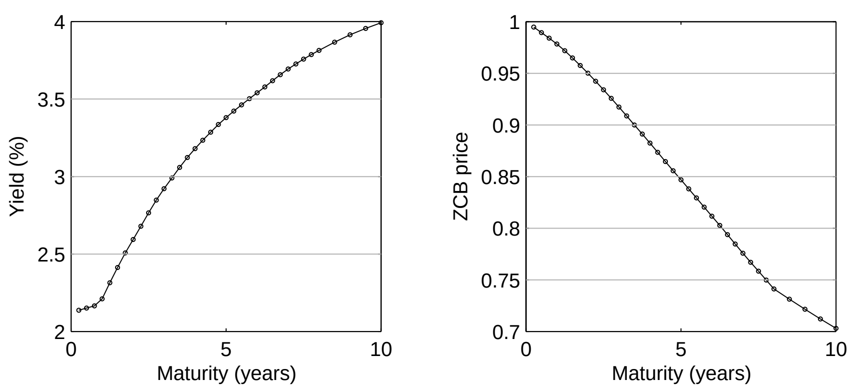

Figure 5. Term structure. The figure on the left displays the zero spot rates for different maturities and the figure on the right depicts the corresponding prices of zero coupon bonds on March 25, 2005.

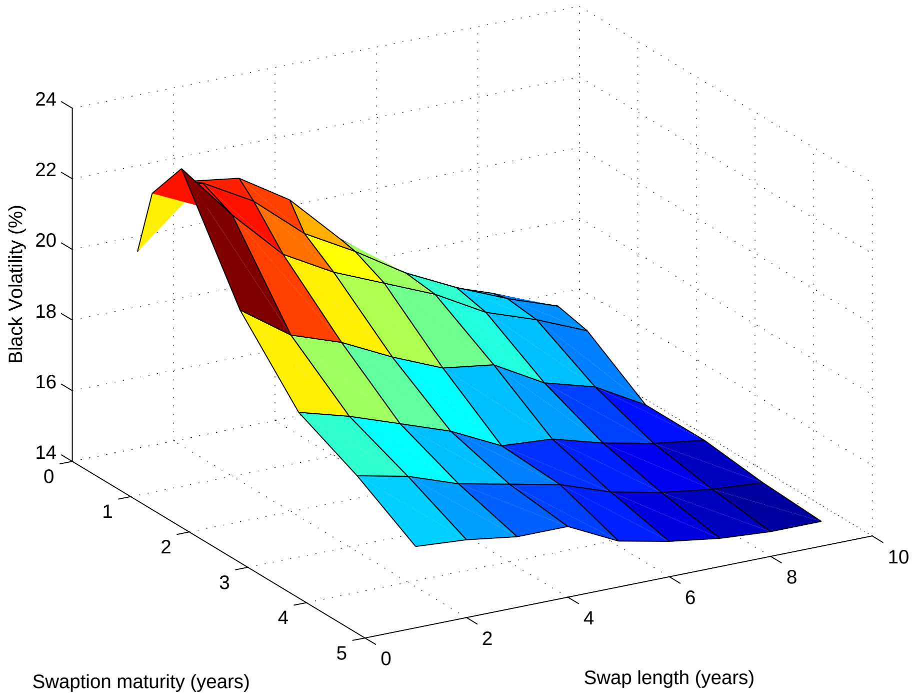

Figure 6. Swaption surface. Implied Black swaption volatilities for swaptions across 3-, 6-month, 1-, 2-, 3-, 4-, 5-year maturities and 1-, 2-, 3-, 4-, 5-, 6-, 7-, 8-, 9-year swap lengths on March 25, 2005.

- December 30, 2005 is depicted in the box-plot in Figure 8. Trading in cap and swaption volatilities is predominantly based on mean-reversion. When volatilities, and other related series such as spreads between cap and swaption volatilities of different maturities and expiries, move away from their historical averages traders bet that they will return. Figure 8 displays several outliers in terms of high volatilities that might have been used as starting points for trading strategies based on mean reversion in volatility levels.

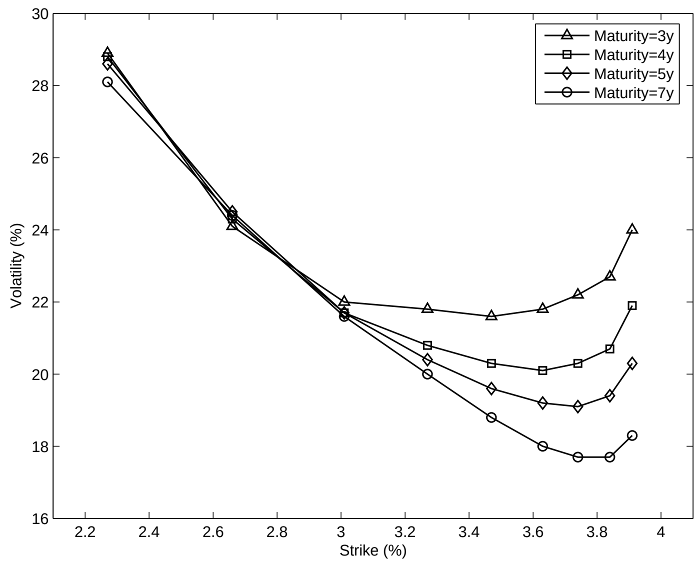

Figure 7. Cap skew. Implied Black volatilities for caps 3-, 4-, 5-, and 7 year maturities and strike rates of 2.27%, 2.66%, 3.01%, 3.27%, 3.47%, 3.63%, 3.74%, 3.84%, and 3.91% on March 25, 2005.

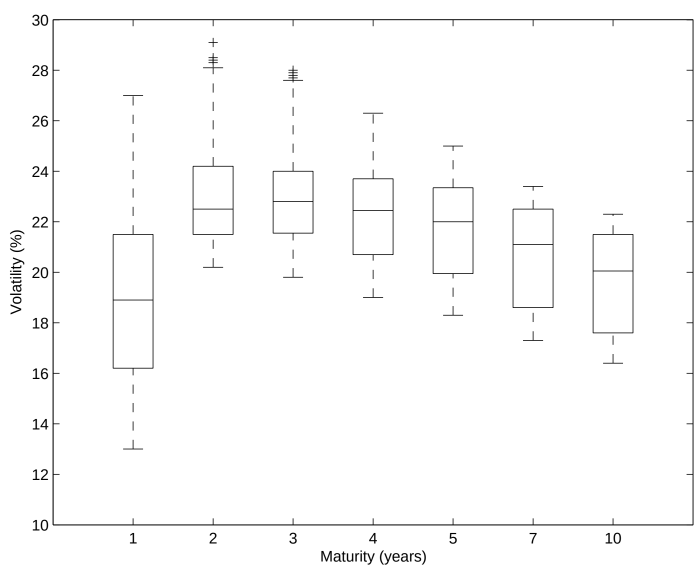

Figure 8. Cap descriptive statistics. Boxplot of cap volatilities for the sample January 4 - December 30, 2005.

VI. Results

This section presents the calibration results. Further, the accuracy for in- and out-of-sample swaption pricing is reviewed. Finally, the ability to predict the future plain-vanilla swaption prices using the Rebonato approximation of the BGM-model is assessed.

A. Calibration

The values of the parameters of the volatility function a, b, c, and d from the calibration for each day in the sample is depicted in Figure 9. Summary statistics are presented in Table V. The evolution in time of the volatility term structure on March 25, 2005 is depicted in Figure 10. The evolution of the volatility term structure is the current caplet volatility term structure and the future caplet volatility term structures implied by the model today. The same figure also shows the calibrated η_i parameters for that day.

Table V
Descriptive statistics for calibrated model parameters

The table presents the average, median, minimum, and maximum values for the model parameters that were obtained when calibrating the model over each day in the sample period. The sample consists of 256 trading days of daily data, from January 4 to December 30, 2005.

	Parameter				
	a	b	c	d	β
Mean	0.51	2.21	0.20	-0.12	0.05
Median	0.55	2.06	0.20	-0.15	0.06
Min	0.03	0.29	0.12	-0.26	0.01
Max	1.00	5.00	0.27	0.09	0.13

When reducing the full-factor model to a two-factor model the parameterization of the correlation structure must be calibrated to the correlation surface implied by the correlation parameter β in the initial calibration. The results of this factor reduction procedure is depicted in Figure 11.

The time evolution of the correlation parameter β and the location of the hump of the volatility term structure given by $(a - bd)/ab$ is depicted in Figure 12.

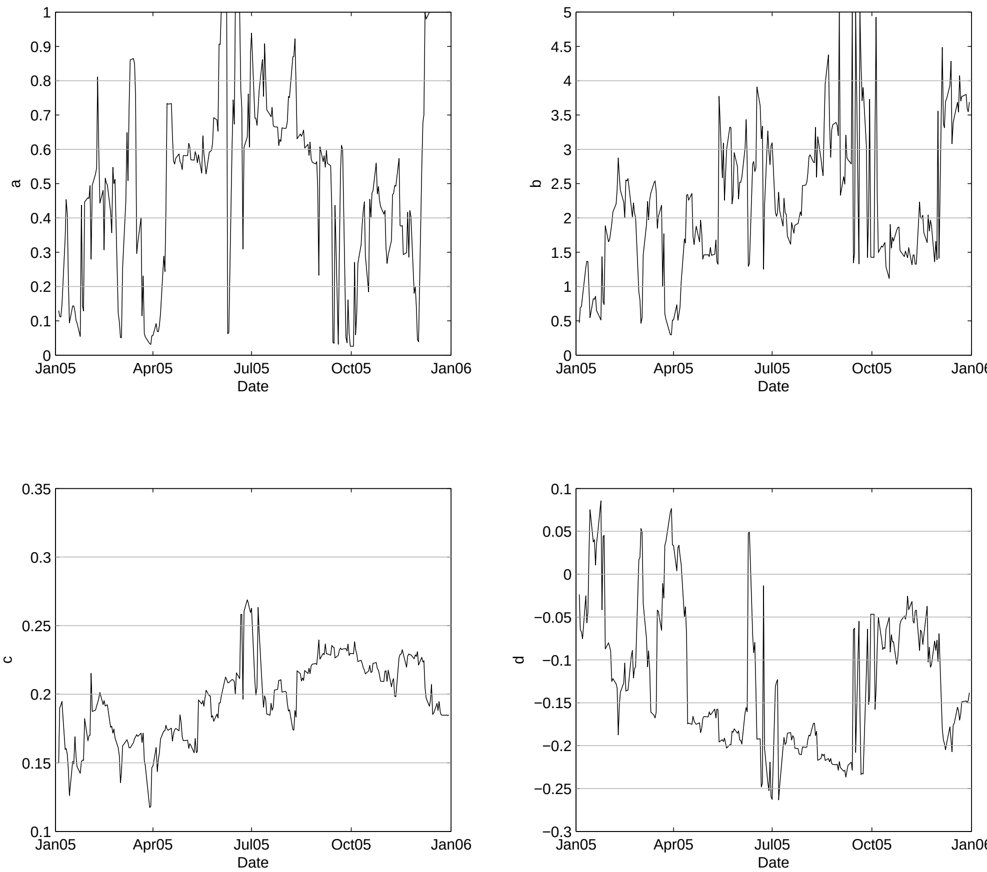

Figure 9. Time evolution of the calibration parameters a, b, c, **and** d**.** The top left picture depicts the parameter a, the top right picture depicts the parameter b, the bottom left picture depicts the parameter c, and the bottom right picture depicts the parameter d.

B. Pricing performance

Table VI presents the results of the pricing error analysis. The average errors and the average absolute errors are reported in volatility points, using Black implied volatility. When comparing swaption pricing on the day of calibration, the average pricing error is 0.08 volatility points and the average absolute error is 0.51 volatility points for swaptions. This is in the same magnitude as the pricing errors obtained in the analysis by De Jong, Driessen, and Pelsser (2001). Hence, the calibration results seem to be reasonably accurate. Furthermore, comparing in-

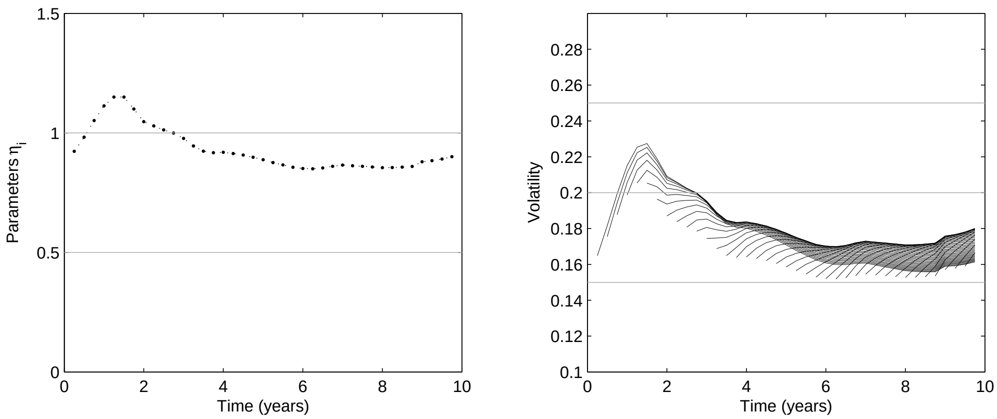

Figure 10. Calibrated η parameters and volatility term structure. The figure shows the η_i parameters of the volatility function (left) and evolution of the volatility term structure (right) resulting from calibration to market data on March 25, 2005.

sample performance, averaging the errors over only the swaptions included in the calibration, with out-of-sample performance, averaging the errors only over the swaptions *not* included in the calibration, it is clear that the difference between the average errors, 0.03 vs. 0.10, as well as the difference between the average absolute errors, 0.47 vs. 0.53, are small in both absolute and relative terms.

When predicting swaption prices one and five days ahead all full-sample error statistics increase slightly; for the full-factor model the average absolute error increases from 0.51 to 0.54 looking one day ahead, and to 0.61 looking five days ahead. Similarly, for the two-factor model, the average absolute error increases from 0.79 to 0.80 and 0.85 respectively.

Looking at the pricing errors grouped by tenor, it can be seen that the average absolute errors are the largest for short tenors and decrease for longer tenors. The 1 year tenor is the most extreme with an average absolute error of 1.16 volatility points compared to 0.54 for the whole sample. This phenomenon is consistent for both the full factor and the two factor model, and for both the one-day and the five-day look-aheads. Considering the average errors, the full-factor model underestimates the prices of swaptions with long tenors and overestimates prices

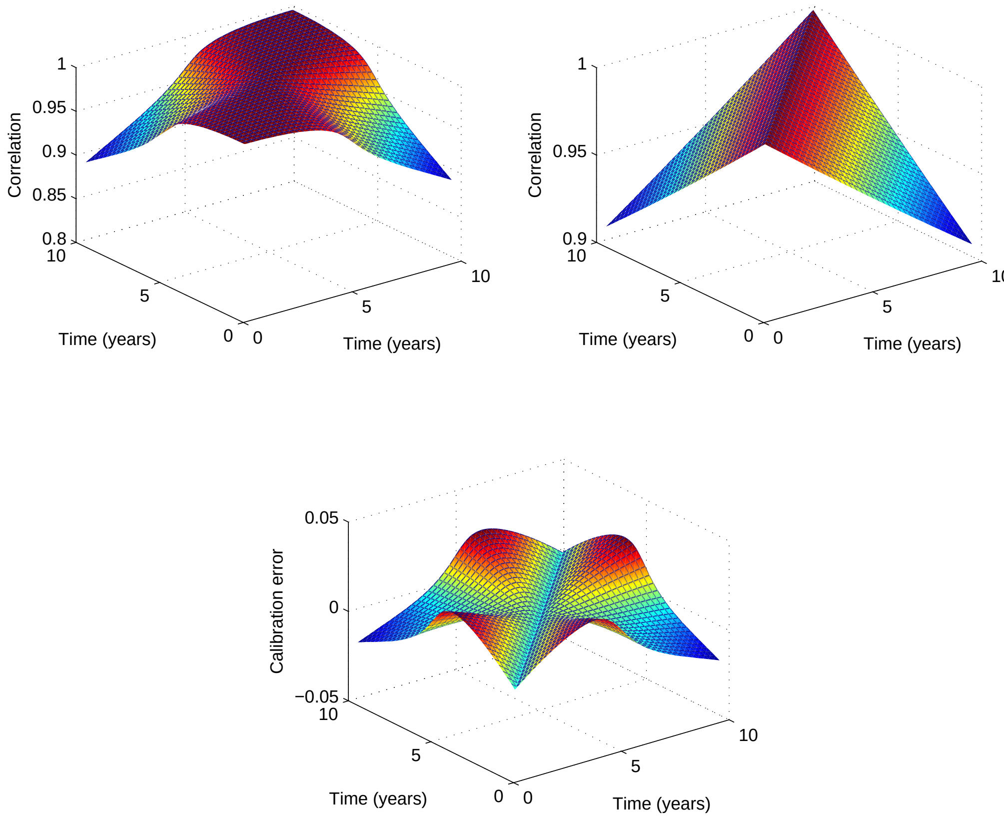

Figure 11. Correlation surface calibration on March 25, 2005. The top left picture depicts the two-factor correlation surface obtained from the full-factor correlation surface, the top right picture depicts the full-factor correlation surface, and the bottom picture depicts the difference between the two surfaces – the calibration error.

of swaptions with short tenors. The two-factor model consistently overestimates swaption prices for all tenors.

Grouping by expiries, it is evident from the average errors that prices for short expiries are underestimated while prices for long expiries are overestimated by both models and for both look-ahead horizons. Furthermore, the average absolute errors are largest for the shortest and the longest expiries. The two-factor model has the most trouble with the three-month expiry, where the average error is -0.92 compared to 0.26 for the whole sample.

As noted in the section on methodology, the systematic biases of the pric-

Table VI
Summary statistics of pricing errors

The table shows measures of the performance of the pricing of swaptions by the BGM model calibrated to caps and swaptions. Results are shown for a full-factor model as well as a two-factor model. The average error is defined as the volatility implied by the model minus the volatility implied by the market, averaged over all days in the sample, and the parts of the swaption matrix as given by the evaluation setting. The average percentage error is calculated as the average error divided by the market implied volatility and the average absolute percentage is calculated similarly.

	Full-factor				Two-factor			
Evaluation setting	Avg error (pts)	Avg abs error (pts)	Avg % error	Avg % abs error	Avg error (pts)	Avg abs error (pts)	Avg % error	Avg % abs error
Same day								
All data	0.08	0.51	0.40	2.56	0.25	0.79	1.31	3.88
Calibrated	0.03	0.47	0.16	2.34	0.25	0.76	1.33	3.75
Other	0.10	0.53	0.51	2.66	0.25	0.80	1.30	3.95
One day ahead								
All data	0.09	0.54	0.45	2.69	0.26	0.80	1.36	3.95
By tenor:								
1 yr	1.00	1.16	4.88	5.61	0.36	1.54	1.76	7.32
2 yr	0.40	0.60	1.98	2.90	0.34	1.03	1.82	4.78
3 yr	0.05	0.53	0.36	2.56	0.30	0.82	1.56	3.94
4 yr	-0.16	0.51	-0.74	2.50	0.19	0.75	0.98	3.66
5 yr	-0.28	0.50	-1.36	2.52	0.17	0.69	0.89	3.50
6 yr	-0.20	0.39	-1.00	2.03	0.20	0.57	1.05	2.98
7 yr	-0.13	0.30	-0.67	1.64	0.24	0.50	1.26	2.65
8 yr	-0.10	0.28	-0.53	1.52	0.24	0.46	1.28	2.47
9 yr	-0.05	0.28	-0.20	1.58	0.28	0.45	1.52	2.53
By expiry:								
1/4 yr	-0.15	0.69	-0.54	3.33	-0.92	1.16	-4.16	5.42
1/2 yr	-0.18	0.50	-0.89	2.40	0.30	0.61	1.39	2.91
1 yr	0.01	0.45	-0.07	2.09	0.45	0.66	2.05	3.14
2 yr	0.04	0.35	0.10	1.83	0.46	0.64	2.24	3.26
3 yr	0.40	0.55	1.90	2.80	0.72	0.80	3.59	4.07
4 yr	0.41	0.57	2.05	3.03	0.73	0.82	3.73	4.29
5 yr	0.49	0.64	2.56	3.47	0.82	0.90	4.31	4.82
Five days ahead								
All data	0.12	0.61	0.59	3.04	0.28	0.85	1.49	4.19
By tenor:								
1 yr	1.02	1.19	5.01	5.74	0.38	1.56	1.89	7.38
2 yr	0.44	0.70	2.16	3.34	0.37	1.08	1.99	5.01
3 yr	0.08	0.59	0.50	2.86	0.32	0.87	1.70	4.18
4 yr	-0.14	0.58	-0.60	2.84	0.22	0.80	1.11	3.94
5 yr	-0.25	0.57	-1.23	2.87	0.19	0.75	1.00	3.82
6 yr	-0.18	0.47	-0.87	2.45	0.22	0.63	1.16	3.30
7 yr	-0.10	0.39	-0.55	2.11	0.26	0.55	1.37	2.91
8 yr	-0.08	0.36	-0.40	1.97	0.26	0.50	1.40	2.69
9 yr	-0.02	0.36	-0.05	2.00	0.30	0.49	1.65	2.71
By expiry:								
1/4 yr	-0.11	0.80	-0.35	3.87	-0.89	1.21	-3.98	5.67
1/2 yr	-0.15	0.60	-0.73	2.84	0.32	0.67	1.53	3.22
1 yr	0.04	0.53	0.06	2.48	0.47	0.71	2.16	3.38
2 yr	0.06	0.40	0.21	2.11	0.48	0.67	2.34	3.45
3 yr	0.42	0.60	2.01	3.03	0.74	0.84	3.68	4.25
4 yr	0.43	0.61	2.18	3.25	0.75	0.85	3.85	4.49
5 yr	0.51	0.67	2.69	3.64	0.84	0.93	4.43	4.99

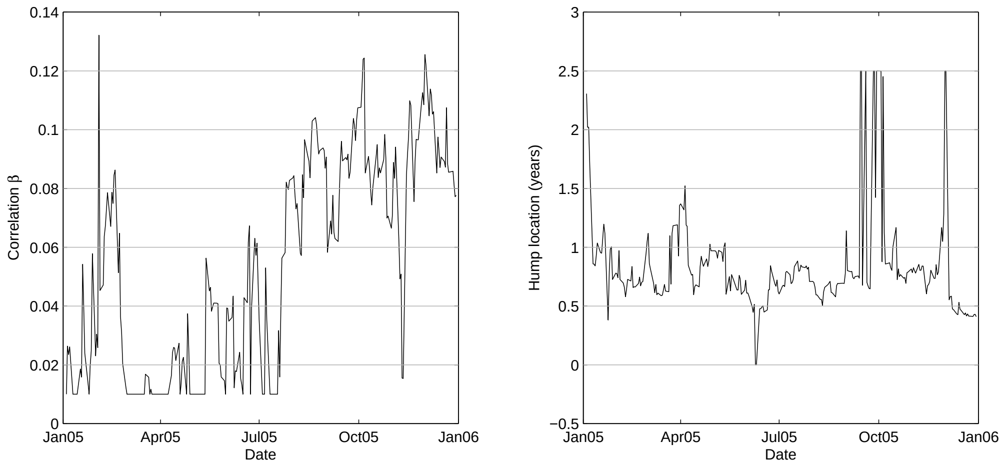

Figure 12. Time-evolution of correlation parameter β and hump location. The left picture depicts the time evolution of the correlation calibration parameter β. The picture on the right depicts the time evolution of the hump location of the volatility term structure $(a - bd)/ab$.

ing models are studied by estimating the following regression, as used by Gupta and Subrahmanyam (2001), for the implied volatility of swaption prices from the market and as estimated by the models:

$$IV_t^{Market} = \beta_0 + \beta_1 IV_t^{Model} + \epsilon_t.$$

The results of the estimation are presented in table VII. The main hypotheses tested are whether β_0 is equal to 0 and whether β_1 is equal to 1. Due to the existence of autocorrelation in the model errors, the standard deviations of the coefficients were estimated using the Newey and West (1987) procedure that accounts for autocorrelation as well as heteroscedasticity. The hypotheses are tested at the 5 % significance level.

For the model evaluation on the calibration day, all evaluation settings results in rejecting both tested hypotheses. Thus, there is a statistically significant bias in the linear relationship between model and market volatilities. β_0 is positive, which implies that there is a systematic underpricing by the models. However, β_1 is less than one, implying the opposite – that the model volatilities have to be reduced by a factor to fit market prices. Combining these two observations, the total bias is that for small values of the volatilities, there is a systematic underpricing by the models, while for larger values, there is a systematic overpricing. This phenomenon is evident for both the full-factor and the two-factor models, but is more severe for the two-factor model. In addition, differentiating the in-sample and out-of-sample data, the phenomenon is evident in both samples, but more so in the out-of-sample data. Figure 13 illustrates this bias, where the left plot shows a grouping of data, all swaptions with expiry of three years, where the model underestimates model implied volatilities lower than 17.4, which is the intersection between the regression line and the perfect model line, and overestimates model implied volatilities above that level. The plot to the right shows a grouping if data, all swaptions with an underlying swap tenor of two years, where the fit is close to optimal.

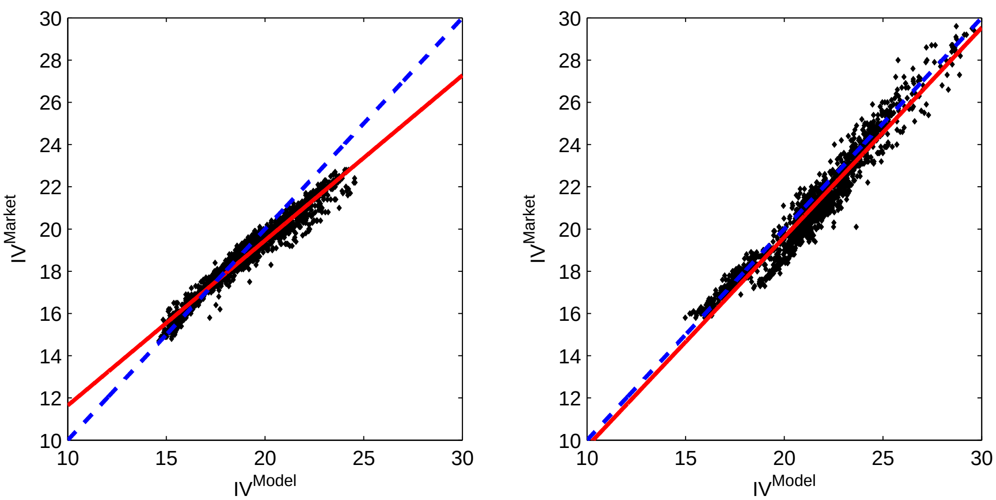

Figure 13. Scatter plot of model volatilities against market volatilities. Market implied volatilities are plotted against model implied volatilities. The solid lines are ordinary least square fits to the data sets, as presented in Table VII. The dashed lines represent a perfect model fit, with $\beta_0 = 0$ and $\beta_1 = 1$. Both plots are for the full-factor model with a one-day look-ahead. The left plot shows data from swaptions with an expiry of 3 years, with regression coefficients $\beta_0 = 3.82$ and $\beta_1 = 0.78$. The right plot shows data from swaptions with tenor of 2 years, with the regression coefficients $\beta_0 = -0.22$ and $\beta_2 = 0.99$.

Considering the one and five day look-ahead the results show the same bias, but to a slightly larger extent. When considering each expiry separately, however, the full-factor model does not show this bias, except for the shortest expiry. It is not possible to reject the hypotheses given above for any expiry except the shortest, and the constant term for the seven year expiry. For the two-factor model things are not as fortunate. Both the constant and the slope coefficients can be rejected for all expiries. Furthermore, the shortest expiry has an R^2 of only 65 %, compared to 83 % for the whole sample. Moreover the constant β_0 is estimated to 6.78 which can be compared to 2.54 for the full sample, and with zero as the optimal value.

Grouping by expiry yields a picture similar to that of the one described for the calibration day. The hypotheses can be rejected for all expiry groups except the shortest expiry of three months. Considering the low R^2 value for that regression, the non-rejection could be attributed to large insecurity in the estimation of the coefficients rather than a god model fit. The remaining expiries show a large amount of bias as described above.

C. Assessment of model robustness over time

In order to evaluate the robustness of the model calibration, the look-ahead time frame was extended from the one-day and five-day windows analyzed in depth earlier up to 150 days. The R^2 of the regression between model and market implied prices is used as a simple performance measure. As can be seen in Figure 14, there is a stable decrease in model performance as the time frame is extended.

The average absolute error for each look-ahead horizon is also calculated and shown in Figure 14. As expected, the average absolute error increases as the horizon is increased.

Table VII
Summary statistics of regression results

This table presents an assessment of the performance of the BGM model calibrated to caps and swaptions. Systematic biases in full-factor and two-factor models are examined by performing the regression $IV_t^{Market} = \beta_0 + \beta_1 IV_t^{Model} + \epsilon_t$. The regression is performed on the data sets conforming to the evaluation settings below. Coefficients estimated not to significantly differ from 0 (1) for β_0 (β_1) at the 5% level are tagged with an $*$. Standard deviations used in these tests are Newey-West adjusted to account for autocorrelation and heteroscedasticity.

	Full-factor			Two-factor		
Evaluation setting	β_0	β_1	R^2 [%]	β_0	β_1	R^2 [%]
Same day						
All data	1.37	0.93	92.3	2.36	0.87	84.1
Calibrated	0.77	0.96	94.6	1.19	0.93	87.3
Other	1.67	0.91	91.2	2.91	0.84	82.6
One day ahead						
All data	1.48	0.92	91.7	2.45	0.86	83.7
By tenor:						
1 yr	1.20	0.90	86.9	6.78	0.67	64.9
2 yr	−0.22*	0.99*	94.3	1.31	0.92	80.2
3 yr	−0.25*	1.01*	93.2	1.15	0.93	84.9
4 yr	0.03*	1.01*	92.1	2.15	0.88	84.5
5 yr	−0.18*	1.02*	93.2	1.81	0.90	87.0
6 yr	−0.21*	1.02*	94.6	1.69	0.90	89.9
7 yr	0.56	0.98*	94.9	2.39	0.86	91.7
8 yr	0.44*	0.98*	95.0	2.26	0.86	92.6
9 yr	−0.36*	1.02*	94.4	1.55	0.90	92.3
By expiry:						
1/4 yr	0.70*	0.97*	86.6	1.43	0.97*	70.5
1/2 yr	1.62	0.93	93.7	1.85	0.90	93.1
1 yr	2.20	0.89	95.9	2.23	0.87	95.1
2 yr	2.70	0.86	96.4	4.00	0.77	94.7
3 yr	3.82	0.78	95.6	4.16	0.75	96.3
4 yr	3.96	0.77	93.9	4.35	0.73	95.2
5 yr	3.78	0.77	91.2	4.19	0.73	93.0
Five days ahead						
All data	1.77	0.91	89.9	2.70	0.85	82.2
By tenor:						
1 yr	1.31	0.89	85.4	6.84	0.66	64.0
2 yr	0.02*	0.98*	92.3	1.53	0.91	78.5
3 yr	0.13*	0.99*	91.1	1.46	0.91	83.1
4 yr	0.68*	0.97*	89.1	2.62	0.86	82.2
5 yr	0.40*	0.99*	90.5	2.22	0.88	84.9
6 yr	0.40*	0.99*	91.6	2.11	0.88	87.6
7 yr	1.25	0.94	91.3	2.86	0.83	88.9
8 yr	1.17	0.94	91.3	2.73	0.84	89.8
9 yr	0.43*	0.98*	90.5	2.04	0.87	89.4
By expiry:						
1/4 yr	1.31	0.94	82.8	2.08	0.94*	67.2
1/2 yr	1.96	0.91	91.2	2.18	0.88	90.6
1 yr	2.36	0.88	94.6	2.39	0.87	93.7
2 yr	3.06	0.84	94.8	4.25	0.76	93.2
3 yr	4.00	0.77	94.7	4.30	0.74	95.5
4 yr	4.08	0.76	93.1	4.43	0.73	94.6
5 yr	3.83	0.77	90.8	4.21	0.73	92.8

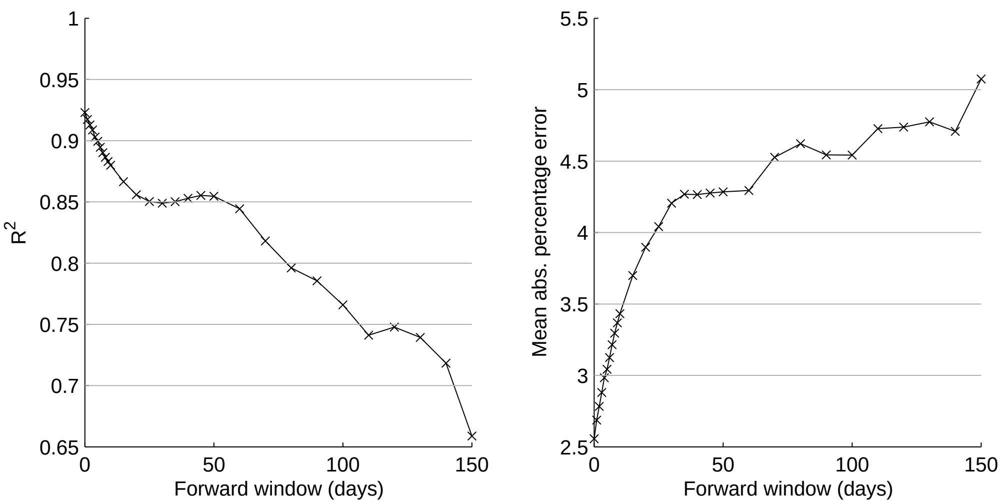

Figure 14. R^2 **and average absolute error as a function of prediction horizon.** The figure shows the performance of the model measured as R^2 of the regression $IV_t^{Market} = \beta_0 + \beta_1 IV_t^{Model} + \epsilon_t$ as the look-ahead time window is increased (left). Also, the average absolute pricing error calculated as the difference between the implied volatility of the market price and the implied volatility of the model price as the look-ahead time is increased (right).

VII. Discussion

This section discusses the obtained BGM-model calibration accuracy. Further, the purpose of the thesis is addressed where the ability of the BGM-model to determine current and predict future plain-vanilla swaption prices is discussed.

A. Calibration accuracy

An empirical study requires clean input data and the importance of this point cannot be emphasized strongly enough. A limited set of swaptions in the swaption matrix were excluded in the sample due to stale data as depicted by the empty entries in Table IV. In total, only two of the excluded swaptions were in the columns in the swaption matrix corresponding to the tenors included in the calibration. Hence, the effects of missing data on the calibration should be small. The calibration of an advanced interest rate model is often time-consuming, even with access to modern day computing resources. The current implementation is, due to its relative simplicity, computationally tractable. A standard PC calibrates the BGM-model according to the non-linear constrained optimization procedure described in the calibration Appendix in approximately 30 seconds.

A.1. Volatility calibration

Figure 10 depicts the time evolution of the volatility term structure and the η_i parameters from the calibration on March 25, 2005. As described in the calibration appendix, the η_i parameters are calibrated exactly to market caplet data and as such they represent the current state of the market at that date. As can be seen in Figure 10, the qualitative shape of the volatility is preserved over time. The calibration accuracy could potentially be improved by fine tuning the calibration to a certain day in the data. Here, where the calibration is performed for a large number of days, such fine tuning is not feasible. The calibration procedure must be automated and scalable. A small adjustment were, however, made for certain days. More precisely, for a few days of data in the sample, the calibration constraints regarding the η_i-parameters were slightly relaxed.

In De Jong, Driessen, and Pelsser (2001) it is argued that a parsimonious model should lead to better predictions for derivatives prices that were not used as input to the calibration. This view is supported in Rebonato (2003) and preferably the calibrated model parameters should be reasonably stable over time. As can be seen in Figure 9 and in Table V, a high variation in the model parameters was obtained.

Further, the bounds on the calibration parameters are hit from time to time. This is especially pronounced for the correlation parameter that tended to zero repeatedly in the first half of the sample. A correlation β tending to zero implies that instantaneous correlations are close to one for all pairs of forward rates, which in turn reduces the model to an one-factor model. More precisely, if all forward rates are perfectly correlated, one stochastic factor controls all the rates fully. This can be seen as a sign of market participants expecting high correlations, but could also be the result of a too restrictive parameterization of the correlation structure, unable to capture the current market correlation environment.

Although the model parameters vary, Figure 12 shows that the location of the volatility hump is relatively stable over time. The location of the hump given by the model is consistent with the empirically observed location at a maturity of between one- and two years.

The high variation of the model parameters suggests that the calibration is sensitive to small changes in market prices of caps and swaptions. The variability of model parameters further suggests that frequent out-of-model hedging is necessary to adjust to the hedge ratios implied by the new model parameters, although the out-of-model hedging is conceptually flawed as argued in Rebonato (2003). The time-variation of model parameters has not been studied in previous research on interest rate derivatives on the STIBOR. In contrast, research has been focused on pricing derivatives using a single day of data, see e.g. Kaisajuntti (2003) and Alpsten (2003).

A.2. Correlation calibration

Figure 11 illustrates how the two-factor parameterization has trouble fully capturing the structure of the initially chosen correlation function in certain regions of the correlation surface. More specifically, the correlation between pairs of nearby forward rates and pairs of distant forward rates are overestimated while the correlation between pairs of one nearby forward rate and one distant forward rate is underestimated. This behavior is discussed by Rebonato (1998) and is believed to stem from the low rank of the two-factor correlation matrix, and can therefore not be circumvented without increasing the number of factors.

The full-factor model outperforms the two-factor model according to essentially all measures. On one hand, this is expected on account of the more restrictive correlation surface of the two-factor model. On the other, there are actually more parameters, one for each forward rate, in the parameterization of the two-factor correlation structure, while only one parameter is used in total to specify the full-factor correlation structure to keep the model as simple as possible. The performance degradation could be the result of the two-step calibration, since it is impossible to perfectly fit the two-factor correlation structure to the simple correlation structure used in the first step of the calibration. If so, it is the cost of a model simplification in order to get more stable results. A full-factor model is feasible when using the closed form pricing formulas, but not for Monte Carlo pricing. Hence, factor reduction is necessary when pricing more complex derivatives where analytic approximations are unattainable.

B. BGM-model swaption pricing performance

Here, the pricing performance of the full-factor and two-factor BGM models on both the day of calibration and days following the calibration is discussed.

B.1. Current date pricing performance

There is an observable difference between in-sample and out-of-sample pricing on the calibration day as seen in Table VI. The BGM-model pricing performance is unanimously worse for out-of-sample swaptions for both model configurations, but the difference is small. This indicates that the calibrated model is parsimonious in the sense that it is not overfitted to the specific data chosen for calibration. Further, a statistically significant bias in relating the model and the market volatilities is observed, as shown in Table VII. More precisely, in the regression model

$$IV_t^{Market} = \beta_0 + \beta_1 IV_t^{Model} + \epsilon_t.$$

the β_0 is found to be significantly different from zero and positive, which implies a systematic underpricing by the models. Simultaneously, however, the results showing a β_1 less than one implies the opposite, namely that the model volatilities need to be reduced by a factor to match the market volatilities. The total bias in current date pricing performance is such that there is a systematic underpricing of the model for small volatilities, while for larger volatilities, there is a systematic overpricing bias. This bias is depicted in the scatter-plot in Figure 13 for swaptions with 3-year expiries and 2-year tenors respectively. The bias is further evident for both the full-factor and the two-factor models, implying that the bias does not primarily stem from a different correlation structures. The bias is also more pronounced for out-of-sample prices than in-sample prices.

B.2. Future date prediction performance

As can be seen in Table VI, the accuracy of the BGM-model predictions of future plain-vanilla swaption prices for one and five day look-ahead horizons is almost as good as the pricing accuracy obtained on the calibration day. There are, however, some apparent biases in addition to the one discussed for the current date pricing performance. Most notably, model performance measured both as the average absolute error the average percentage absolute error is worst for the shortest tenor and decreases with increasing tenor length. This effect is present for both the full-factor and the two-factor models and for both look-ahead horizons. Thus, it seems to be a result of the general model specification rather than choice of correlation structure or number of factors.

Moreover, the two-factor model has severe difficulties with the shortest swaption expiries, with an average error of -0.92 volatility points compared to the total sample average of 0.26 points. This effect is not evident in the full-factor model, which indicates that the problem is a result of the calibration of the two-factor correlation matrix. Additionally, the interpolation method used to extract the caplet volatilities for the shortest maturities may influence the pricing performance of the shortest swaption expiries negatively.

The performance of the prediction using longer look-ahead horizons show clear and, almost, monotonous degradation of model performance as the look-ahead increases as depicted in Figure 14. This implies that recalibration of the model is needed from time to time. The degradation of model performance is not, however, as rapid as the high variability of the calibrated model parameters might lead one to think. Hence, the actual frequency of out-of-model vega-hedging will more likely be determined by a combined view of the degradation in the prediction ability of the model and the amount of transaction costs incurred, and only to a lesser extent take the high variability of model parameters into account.

VIII. Conclusions

In this thesis the ability of a full-factor and a two-factor specification of the BGM-model to price current date and to predict future plain-vanilla swaption prices for a time-series of data is assessed. The main findings are the following:

- The simultaneous calibration to market observed cap and swaption prices yields model parameters with a high temporal variation, implying that out-of-model vega-hedging is necessary.
- The pricing performance on the day of calibration is reasonably accurate, both in absolute terms and in comparison with earlier studies. Out-of-sample tests show that the model does not seem to be overfitted to the data used in the calibration.
- There exist significant biases in the pricing performance, where some sub-samples of the data yields below average results.
- Reducing the full-factor model to a two-factor model decreases the current date pricing performance as well as the future date prediction performance.
- The pricing performance decreases only slightly when using the calibration parameters one and five days after calibration. Over longer time horizons the performance deteriorates gradually.

Finally, the literature on pricing interest rate contingent claims is vast and the level of sophistication in the models used have evolved rapidly during the last few years. The models often include advanced features such as jumps, stochastic volatilities, and this also in a multi-factor setting. The results here indicate that a relatively simple approach might be sufficient and that the real issue when pricing interest rate derivatives should be focused on reaching a good calibration of the model, which is in line with the view of market practitioners (Andreasen 2003).

Suggestions for further research

Further research on the BGM-model swaption pricing model where different model specifications than used in this thesis are applied is suggested. Further, the thesis has focused on the BGM-model, not paying attention to Jamshidian's BGM/J swap market model. Investigating the performance of the swap market model for Swedish interest rate data is a natural extension and complementary approach to the topic addressed in the thesis. Moreover, the incorporation of the volatility skew into the BGM-framework has recently received a lot of attention. Extending the work done in this thesis by assessing the implications of incorporating the volatility skew in the BGM-framework for predicting future plain-vanilla swaption prices is a venue open for further research. Finally, the Rebonato formula used to approximate swaption prices in the BGM-model can be replaced by more advanced formulas such as the one suggested by Kawai (2003). Assessing the improvement in calibration and pricing performance using that formula would be an interesting item of research.

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A. Implementing the BGM-model

In the section on methodology the problem of calibrating the BGM-model was outlined. Here the calibration procedure is described in detail.

A. Volatility - calibration to cap prices

In this thesis, a calibration to market implied volatilities and correlations is chosen. Hence, the calibration procedure estimates the parameters of the instantaneous volatilities $\sigma_i(t)$ and the instantaneous correlations $\rho_{ij}(t)$ for $i, j = 1, \ldots, M$ from market observed cap and swaption implied volatilities. The BGM-model forward rate dynamics under the forward measure Q_i is then given by

$$dF_i(t) \quad = \quad \sigma_i(t)F_i(t)dW_i(t), \quad t \leq T_{i-1} \tag{7}$$

where $dW_1, \ldots, dW_M$ are Brownian motions with pairwise correlations

$$\mathrm{E}[dW_i(t)dW_j(t)] \quad = \quad \rho_{ij}(t)dt. \tag{8}$$

By definition, a cap is split additively into caplets, each depending on a single forward rate. Hence, cap prices are independent of the instantaneous correlations ρ_{ij} between forward rates. To price caps with the BGM-model it is thus sufficient to calibrate the model to cap prices. When pricing swaptions, the correlation between forward rates needs to be accounted for, and thus the calibration is done for caps and swaptions simultaneously. Further, a calibration to both cap and swaption prices is believed to yield a parsimonious model with a robust evolution of the term structure of volatility (Brigo and Mercurio 2001). The model is calibrated by choosing the model volatility function σ_i to exactly fit the square of the market implied caplet volatilities σ_i^{Black} which yields

$$(\sigma_i^{Black})^2 T_{i-1} = \int_0^{T_{i-1}} \sigma_i^2(s)ds \tag{9}$$

for the caplet paying the difference between the T_i-maturity spot rate reset at time T_{i-1} and a strike rate R, if this difference is positive, and zero otherwise.

Several specifications for the choice of the volatility term structure have been proposed. The two most common approaches are based on either using step-wise and flat market volatilities or by using a richer parametric structure. Here, as proposed in Alexander (2002), a parametric specification of the forward rate volatility that accounts for mean reversion is used. A refined multi-parameter formulation of the volatility is subject to the risk of over-fitting the data. The calibration to a parametric volatility structure where the forward rates are assumed to be time-varying but deterministic is carried out as follows. A popular parameterization of the volatilities $\sigma_i(t) = \eta_i h(t)$, where the time-varying parameter $h(t)$ has a hump shaped form and is common to all volatilities is used. The hump shaped form of $h(t)$ was introduced in Rebonato (1999b) and the model volatility reads

$$\sigma_i^{Model}(t) = \eta_i h(t) = \eta_i \left((a(T_{i-1} - t) + d)e^{-b(T_{i-1}-t)} + c \right). \tag{10}$$

The parameters η_i ensures that the level of the instantaneous volatilities can be adjusted to coincide with the level of the market observed cap prices.

B. Correlation - calibration to swaption prices

To price swaptions, that in contrast to caps and floors also depend on the correlation between forward rates, a correlation structure for the BGM-model needs to be chosen. The complete instantaneous correlation matrix ρ consists of $M(M-1)/2$ parameters, where $M = \beta - \alpha$ is the number of forward rates in the swap rate underlying the swaption. To have a parsimonious model, the number of parameters in the matrix needs to be reduced. A simple, full-factor specification can be based on the assumption of an instantaneous correlation matrix of the form $\rho_{i,j} = e^{-\beta|T_i - T_j|}$, see e.g. Kaisajuntti (2003).

The number of parameters can be substantially reduced, by setting $dW(t) = BdZ(t)$ where $Z(t)$ is a n-dimensional Brownian motion with $n \ll M$. Hence, the

$M \times n$ matrix B of rank n, such that $\rho^B = BB'$ is a n-rank correlation matrix, needs to be defined. For plain-vanilla swaptions, Brigo and Mercurio (2001) note that a three factor correlation structure, i.e. with $n = 3$, does not significantly improve the calibration over a two-factor structure. Hence, the two-factor correlation structure suggested in Rebonato (1999b) is used. It should be noted, however, that the correlation structure chosen by practitioners is closely tied to the specific option priced. For pricing strongly correlation-dependent options more advanced correlation calibration methods are necessary, see e.g. Schoenmakers and Coffey (2003). The two-factor approach in Rebonato (1999b) consists of M parameters $\theta_1, \ldots, \theta_M$ where M is the number of forward rates. All the M variables are kept free for calibration. Specifically, the entries of the matrix B and the correlation matrix ρ for the forward rates are given by

$$b_{i,1} = \cos\theta_i, \quad b_{i,2} = \sin\theta_i, \quad \rho^B_{i,j} = b_{i,1}b_{j,1} + b_{i,2}b_{j,2} = \cos(\theta_i - \theta_j) \tag{11}$$

for $i = 1, \ldots, M$ where $\theta_1, \ldots, \theta_M$ are parameters determined from the calibration to observed swaption prices.

C. The Rebonato formula

Since log-normal forward rates are assumed in the BGM-model, swaption prices have to be approximated (Brigo and Mercurio 2001). Several approximative solutions to swaption prices in the BGM-framework have been developed. In practice, either a Monte Carlo simulation or an analytical formula is used. Among the existing analytical formulas the Rebonato formula, the Hull and White formula, an approximation assuming that the volatility of the swap rate can be viewed as a weighted sum of forward rate covariances (Rebonato and Jäckel 2000), the Brace rank-one or rank-r formula (Brigo and Mercurio 2001), or the formula based on an asymptotic series expansion proposed by Kawai (2003) are common. The Rebonato approximation is the zeroth order term in a series expansion where moderate noise is assumed. The asymptotic expansion proposed by Kawai (2003), which includes higher order expansion coefficients, is reported to increase the accuracy of the calibration. Assessments of the accuracy of the different approaches have showed that Rebonato's approximation is sufficiently accurate, see Brigo and Mercurio (2001) and Brigo, Mercurio, and Morini (2005). Hence, the Rebonato approximation formula is used to calibrate the BGM model parameters to market observed swaption prices.

The Rebonato formula uses that the forward swap rate can be considered as a linear combination of forward rates under the assumption that forward rates have a log-normal distribution. In the formula, the BGM-model Black-like swaption volatility is approximated as

$$(\sigma_{\alpha,\beta}^{BGM/Rebonato})^2 \approx \frac{1}{T_\alpha} \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0)w_j(0)F_i(0)F_j(0)\rho_{i,j}}{S_{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)dt \quad (12)$$

More precisely, the forward swap rate $S_{\alpha,\beta}(t)$ can be written as

$$S_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} w_i(t)F_i(t) \quad (13)$$

where

$$w_i(t) = \frac{\tau_i FP(t,T_\alpha,T_i)}{\sum_{k=\alpha+1}^{\beta} FP(t,T_\alpha,T_k)} = \frac{\tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j F_j(t)}}{\sum_{k=\alpha+1}^{\beta} \tau_k \prod_{j=\alpha+1}^{k} \frac{1}{1+\tau_j F_j(t)}} \quad (14)$$

and

$$FP(t,T_\alpha,T_i) = P(t,T_i)/P(t,T_\alpha) = \prod_{j=\alpha+1}^{i} FP_j(t), \quad FP_j(t) = \frac{1}{1+\tau_j F_j(t)} \quad (15)$$

denotes the "forward discount factor". By noticing that the variability of the w's is much smaller than the variability of the F's and thus assuming constant $w_i(t)$, i.e. by setting

$$S_{\alpha,\beta}(t) \approx \sum_{i=\alpha+1}^{\beta} w_i(0)F_i(t) \quad (16)$$

the Rebonato formula can be derived (Brigo and Mercurio 2001).

D. Calibration procedure

The calibration procedure for a simultaneous calibration to market observed implied volatilities and correlations can be summarized as follows. First, the η_i's are obtained as functions of the parameters a, b, c, and d by calibrating to market observed caplet volatilities σ_i^{Black} by the relation

$$\eta_i^2 = \frac{(\sigma_i^{Black})^2 T_{i-1}}{\int_0^{T_{i-1}} h^2(T_{i-1}-t)dt}. \tag{17}$$

The integral of the square of the parametric model volatility $h(t)$ can be evaluated using a software for formal manipulation, e.g. MAPLE. Second, a set of European swaptions are priced with Rebonato's swaption formula which yields the BGM-model swaption prices as functions of a, b, c, d, and the θ's. The model swaption price will thus depend on the correlation parameters $\theta_1, \ldots, \theta_M$. The parameters are then chosen to match the market observed swaption prices chosen for calibration as closely as possible. This is achieved by minimizing the sum of squared differences between model and market observed swaption prices according to

$$\min \sum_{\alpha,\beta} \left(\sigma_{\alpha,\beta}^{BGM/Rebonato} - \sigma_{\alpha,\beta}^{Black}\right)^2 \tag{18}$$

subject to

$$0 < a \leq 0.5, \quad 0 < b \leq 5, \quad 0 < c \leq 0.5, \quad -1 \leq d \leq 1 \tag{19}$$

$$c + d \geq 0, \quad 0 \leq \frac{a - bd}{ab} \leq 6 \tag{20}$$

$$0.01 \leq \beta \leq 10 \tag{21}$$

$$0.85 \leq \eta_i(a, b, c, d) \leq 1.15 \tag{22}$$

The constraints on the parameters a, b, c, and d ensure that the minimization of the sum of squared deviations between model and market observed swaption prices yields a reasonable shape of the volatility function. For small values of $T_{i-1} - t$ in the expression for the volatility, $c + d$ should coincide with the implied volatilities for caps with short maturities. Hence, the constraint $c + d > 0$, imposing non-negative cap volatilities for caps with short maturities, is reasonable. Similarly, for large $T_{i-1} - t$ the value of c must be in line with the implied volatilities of caps with very long maturities. Hence, the requirement $c > 0$ is also reasonable.

The non-linear constraint imposed on the first derivative of the volatility function with respect to $T_{i-1} - t$, i.e. $(a - bd)/ab$, ensures that the location of the extremum of the volatility term structure is not located too far out in the future from today. The constraint $a > 0$ forces the extremum to be a maximum. Further, the constraints on the η_i's ensure that the shape of the term structure is preserved in time, i.e. that the volatility term structure is time-homogeneous. Finally, constraining the correlation parameters to the specified intervals ensures that the correlation does not change too rapidly between forward rates. The above discussion follows Brigo and Mercurio (2001) and Kaisajuntti (2003).

In practice, the state of of the cap market data on a few days in the examined sample made it necessary to relax the constraints on the η_i's in order for the minimization procedure to find a feasible solution for the volatility parameters. In these cases, the bounds on the η_i's were expanded to:

$$0.7 \leq \eta_i(a, b, c, d) \leq 1.35. \tag{23}$$

In the factor reduction process, the sum of squares of differences between correlations is minimized:

$$\min \sum_{i,j} \left(\rho_{i,j}^{full-factor} - \rho_{i,j}^{two-factor} \right)^2 = \min \sum_{i,j} \left(e^{-\beta |T_i - T_j|} - \cos(\theta_i - \theta_j) \right)^2 \tag{24}$$

subject to

$$-\pi/3 \leq \theta_i - \theta_{i-1} \leq \pi/3, \quad 0 \leq \theta_i \leq \pi \tag{25}$$

B. Mathematical Background

This appendix outlines the mathematics underlying the BGM-model. A treatment of the topics discussed with emphasis on the mathematics can be found in Björk (2004) and with emphasis on implementation in Joshi (2003).

A. Tools used in the BGM-model derivation

The concept of a *martingale* captures the essence of a fair game. It can be characterized as a stochastic process that has no growth on average. More precisely, a martingale is a stochastic process X with the property that the expected future value of the process equals the value of the process today or more precisely that

$$\mathbf{E}_t[X(s)] \quad = \quad X(t) \qquad \forall s \geq t. \tag{26}$$

A *numeraire* is a way of expressing prices. The money market account expresses prices in monetary units like e.g. SEK, but one is also free to choose a numeraire that expresses prices in e.g. the number of shares or in a foreign currency. A numeraire must be a traded asset with a positive price. Let $W(t)$ denote a standard Brownian motion. *Girsanov's* theorem defines the effect of a change of measure on a Brownian motion. Given a Brownian motion with drift $dW' = dW - \mu dt$ and facing the problem of finding a probability measure $\mathbf{P}'$ such that W' is a martingale, i.e. with zero drift, then W must have drift $+\mu dt$ under the new measure $\mathbf{P}'$. The ratio between two probability measures $d\mathbf{P}'/d\mathbf{P}$ is called the *Radon-Nikodym derivative*. The Radon-Nikodym derivative is a stochastic process that is a martingale under the original measure $\mathbf{P}$. It is used when the expectation under a different measure is required and it can be intuitively seen as the ratio between two different numeraires. For any well-behaved stochastic process $\xi(t)$ the Radon-Nikodym derivative reads

$$\frac{d\mathbf{P}'}{d\mathbf{P}} \quad = \quad \exp\left(\int_0^t \xi(s)dW(s) - \frac{1}{2}\int_0^t \xi(s)^2 ds\right) \tag{27}$$

where the process $dW' = dW - \xi(t)dt$ is a Brownian motion under the measure $\mathbf{P}'$. The *change of numeraire theorem* states that the value of a derivative must be the same if calculated with two different numeraires N and M. This expresses the common sense assumption that the price of a contingent claim should not depend on the units its price is measured in.

B. BGM-model dynamics

The BGM- and BGM/J-models are based on the same main idea; to choose a different numeraire than the risk-free account. Normally, one works under a risk neutral martingale measure. The idea that Brace, Gatarek, and Musiela (1997) and Jamshidian (1997) used was to replace the risk neutral martingale measure with a forward measure, where the numeraires are bond price processes. Using the terminology defined in the previous section, the derivation of the forward rate dynamics in the BGM-model is roughly as follows. Let T_{i-1} be the i:th reset date, $F_i(t)$ the forward LIBOR rate for $[T_{i-1}, T_i]$, $\sigma_i(t)$ the volatility of $F_i(t)$, and $\tau_i = T_i - T_{i-1}$ the tenor. By using that the LIBOR rate is defined from the discount bonds as

$$F_i(t) \equiv \frac{1}{\tau_i}\left(\frac{P(t,T_{i-1}) - P(t,T_i)}{P(t,T_i)}\right) \tag{28}$$

and choosing $P(t, T_{i+1})$ as the numeraire the process F_i is martingale under the measure $\mathbf{Q}^{i+1}$. Thus each forward LIBOR rate F_i is martingale under its own measure and the task is to find one single measure for all F_i:s. The measure is obtained by changing measure by using the Radon-Nikodym derivative which yields

$$\begin{aligned}
\mathbf{R}(t) &= \frac{d\mathbf{Q}^i}{d\mathbf{Q}^{i+1}} \propto \frac{P(t,T_i)}{P(t,T_{i+1})} = 1 + \tau_i F_i(t) && (29)\\
d\mathbf{R} &= \tau_i \sigma_i F_i dW = \frac{\tau_i \sigma_i F_i}{1 + \tau_i F_i}\mathbf{R}dW && (30)
\end{aligned}$$

and hence

$$dW^i = dW^{i+1} - \frac{\tau_i \sigma_i F_i}{1 + \tau_i F_i} dt. \tag{31}$$

Suppose that there are N forward LIBOR rates in total. The final LIBOR rate F_N is martingale under the measure $\mathbf{Q}^{N+1}$. If all LIBOR rates are considered under this measure the dynamics $dF_N = \sigma_N L_N dW^{N+1}$ is obtained. Using the results above and working backwards one obtains

$$\begin{aligned} dF_{N-1} &= \sigma_{N-1} L_{N-1} dW^N & (32) \\ &= \sigma_{N-1} F_{N-1} \left[dW^{N+1} - \frac{\tau_N \sigma_N F_N}{1 + \tau_N F_N} dt \right] & (33) \\ &= -\frac{\tau_N \sigma_N F_N}{1 + \tau_N F_N} \sigma_{N-1} F_{N-1} dt + \sigma_{N-1} F_{N-1} dW^{N+1} & (34) \end{aligned}$$

which for a general LIBOR rate F_i finally gives us the dynamics as

$$dF_i = -\left(\sum_{j=i+1}^{N} \frac{\tau_j \sigma_j F_j}{1 + \tau_j F_j} \right) \sigma_i F_i dt + \sigma_i F_i dW^{N+1}. \tag{35}$$

The above dynamics describes the evolution of each individual forward rate and it allows the vector of forward rates to be evolved by Monte Carlo simulation. The price of interest rate options can then be calculated as the average of the prices obtained over a large number of simulated forward rate paths using the dynamics derived above. Prices can also be approximated using formulas such as the one suggested by Rebonato and Jäckel (2000).