MASTER THESIS IN FINANCE

Skewness in portfolio allocation: a comparison between different mean-variance and mean-variance-skewness investors

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Abstract

We investigate the problem of portfolio allocation using an expected utility framework where investors' preference for positive skewness of returns is introduced. We develop a three-parameter generalized utility function that can be used to capture the full spectrum of absolute and relative risk aversions from CARA to DRRA in comparable settings. We approximate the utility function with the use of a Taylor series expansion truncated at different points to include or exclude the preference for skewness. We then find different optimal mean-variance and mean-variance-skewness portfolios and compare them with each other by looking at their absolute distances in space and differences in certainty equivalent, across investors with different levels of risk aversion and different kinds of risk aversions. We find that the mean-variance and mean-variance-skewness solutions to the portfolio choice problem diverges more as the overall level of risk aversion increases, as well as when investors exhibits utility functions with decreasing relative risk aversion (DRRA) and decreasing absolute risk aversion (DARA). Differences in certainty equivalent between the mean-variance optimization and mean-variance-skewness optimizations can be economically significant for highly risk averse investors and DARA/DRRA investors.

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2 Introduction

The mean-variance analysis proposed by Markowitz (1952) can be unanimously considered to be the backbone of modern financial theory and it plays a central role in the investment management and wealth management practices. In his famed paper, Markowitz proposes a framework in which a rational investor chooses her portfolio by trading-off a high expected return for a low variance of return. Further, such portfolios are to be chosen from a set of different combinations which are “efficient” under a mean-variance framework, meaning that each portfolio belonging to this set exhibits the lowest variance for a given level of expected return, or equivalently, the highest level of expected return for a given variance.

Markowitz’s theory is however severely limited by its assumptions that an investor making a portfolio decision is only concerned about mean and variance of their portfolio return, or, equivalently, by implicitly assuming that financial returns are normally distributed. As it is often the case when modelling real life choices of supposedly rational individuals, there is always an important trade-off between a model’s analytical simplicity and the full representativeness of all the variables that might come into play.

Hence, while the mean-variance framework provides some remarkable intuition for a variety of theoretical and practical purposes, most notably when it comes to the importance of portfolio diversification, in the real world, neither of the two aforementioned assumption holds. It has been empirically shown that financial returns are not normally distributed, and that higher statistical moments come into play to describe how return distributions are shaped. Investors care about higher statistical moments of the distributions of their portfolio returns and they make investment choices based on higher moments as well, not just on mean and variance.

Among these neglected moments, skewness seems to play a prominent role in financial decision making. Skewness is the third central moment that describes the shape of statistical distribution, mean being the first and variance the second. While the mean is a measure of central tendency and variance one of dispersion around the mean, skewness is a measure of asymmetry of a distribution. An asymmetrical, or “skewed” distribution is one whose two sides are not specular around the mean. In a “skewed” distribution, the majority of observations lay on one side of the mean, while the other side balances out with fewer observations whose values are further away from the mean. The skewness of a distribution measures its degree of asymmetry, either leaning towards one side or the other.

It is straightforward to say that a rational risk averse investor has a preference for high expected returns and an aversion to the variance of returns. When it comes to skewness, whose value could be either positive or negative, a risk averse investor will prefer positively skewed returns over negative ones, the more positive the better. It is intuitive to understand why: the more positively skewed the return distribution, the lower the probability of large losses to occur and the higher the probability of realization of extreme positive values. Hence, an asset whose return follows a positively skewed distribution would be preferable to an asset whose returns follow a negative one, as it would present the possibility of very high rewards, while the opposite would apply for the negatively skewed one.

As it is the case in mean-variance analysis, where covariance needs to be taken into account when computing the total variance of portfolio return, co-skewness between different assets' returns, as a measure of the tendency of the extremes of a distribution to be correlated with each other, also plays a fundamental role in mean-variance-skewness portfolio selection.

While the mean-variance problem is a two dimensional one, incorporating skewness into the analysis expands the allocation problem into a third dimension, where the investor now faces to obtain two different competing objectives: maximizing expected return and positive skewness of a portfolio, while minimizing portfolio variance.

The goal of this paper is to investigate, analytically and empirically, whether or not considering the skewness dimension is negligible when making portfolio allocation decisions, and to understand whether different utility maximizing investors chose to allocate their wealth differently when their preference for skewness is taken into account, compared to a simpler mean-variance framework. While this is not an innovative approach per se, as it has been done before, we give our contribution to the existing literature by providing a deeper investigation on the portfolio choice problem with skewness by analysing it from the perspective of different kinds of absolute and relative risk aversion.

In our work, we develop an expected utility framework by deriving a parametric utility function of wealth, generalized to fit different absolute and relative kinds of risk aversion given different parameters. We then approximate the newly derived utility function by mean of a Taylor series expansion, to make it suitable for numerical optimization. We further use a numerical optimizer to find the portfolio that maximizes the expected utility, both in a mean-variance and a mean-variance-skewness framework using different sets of assets. Short selling is not allowed and a riskless asset is made available to the investor. We then compare the portfolios obtained under the MV and MVS numerical optimizations and see that the sets of weights in the two portfolios can be considerably different for some parameters of risk aversion, meaning that failing to consider skewness of return distributions and an investor preference for skewness, might lead to the formation of portfolios which do not yield the optimal level of expected utility for some investors. To further investigate the loss in utility, we also look at the difference in certainty equivalents, i.e. the hypothetical risk-free rate that would yield the same utility to the investor as the corresponding risky portfolio, to investigate economic significance and reasons for extending the framework. Further, we analyse whether and how the Sharpe-ratio yielded by each investor's optimal portfolio changes when skewness is taken into account.

Our analysis is fairly similar to the work of Jondeau and Rockinger (2006) but yet with some considerable differences, allowing us to add a contribution to the existing literature. Jondeau and Rockinger (2006) use the Taylor series expansion to approximate an exponential utility function, to include a preference up to the second, third, fourth moment of the distribution of portfolio returns and then they maximize expected utility as a function of portfolio weights. They see that, when the distribution of returns of the assets under consider shows a large departure from normality, the mean-variance-skewness approximation proves considerably better at maximizing the expected utility, compared to a mean-variance approach. In their paper, they emphasize how an investor would allocate her wealth differently by using a mean-variance-skewness approach instead of a mean-variance approximation, and find that such difference is not relevant when the return distribution of assets under consideration does not show a significant departure from normality, but the difference between the two increases when the departure from normality of the returns' distribution becomes larger and the investor has a larger aversion to risk. Hence, they conclude that the same individual would invest differently when adopting a mean-variance-skewness rather than a mean-variance approach to the portfolio problem, but only when the assets returns' distribution show a large departure from normality.

Our approach is slightly more comprehensive, in a sense that we do not only wish to see how the same optimization would behave across different sets of assets, but how the optimization would change when different kinds of risk aversion are involved. To do so, we developed a parametric utility function that can fit different kinds of relative and absolute risk aversions, as a comparable common base for the different cases under investigation.

We find that, everything else being equal, the greater the overall aversion to risk, the larger the difference between two portfolios optimized via MVS approach, compared to the MV approach, in line with the findings of Jondeau and Rockinger (2006). Additionally, we observe that, for equal levels of overall risk aversion, such difference is the lower for an investor displaying constant absolute risk aversion and the higher for an investor displaying decreasing relative risk aversion.

The fundamental intuition behind our analysis is that the skewness dimension is rather relevant for an investor making a portfolio decision. This can have a considerable impact in financial theory and financial practice. A noticeable portion of financial economics finds its ground on the assumption that rational investors are mean-variance investors, who do not consider skewness when making their choices. It is common practice among wealth management professional to inquiry about clients' personal attitudes to risk before making portfolio recommendations. We argue that neglecting to investigate about clients' preference for skewness and failing to include such preference in their analyses, will yield a different and suboptimal portfolio allocation for an investor with defined preference over skewed returns.

3 Previous literature

Although a considerable portion of financial theory relies, either explicitly or implicitly, upon the assumption that financial returns follow a normal distribution, it has been demonstrated that this is not empirically the case. Mandelbrot (1963) proposes an alternative probability distribution to describe the behaviour of returns, after recognizing that their distributions are unlikely to fit under a Gaussian curve, mentioning a discovery from economist Wesley Clair Mitchell dating back as early as 1915. Fama (1965) confirms the inadequacy of the normality assumption questioned by Mandelbrot, rejecting it while finding evidence of thicker tails than those of the normal distribution. Kon (1984) finds empirical evidence of skewness and kurtosis in 30 stocks in the Dow Jones Industrial Average, as well as in market indexes, while, more recently, Peiró (1999) run empirical test for sample skewness under the assumption of normality on a sample of eight international stock markets and three foreign exchange markets, rejecting their symmetry in all but one case.

The general consensus is that investors prefer positive skewness, all else being equal. Within an expected utility theoretical framework, Scott and Horvath (1980) prove that for a risk averse investor, the $m-th$ derivative of the utility function is positive if m is odd, and negative if m is even, irrespective to the level of wealth. Such property can be used to determine whether there is a preference or aversion to a certain moment of return, as in the application of Taylor's expansion to expected utility. This is consistent with Arditti (1967) confirming that there should be a preference for skewness, at least intuitively. Scott and Horvath (1980) additionally state that mean-variance approach is sufficient if at least one of the following three conditions hold: either the distribution of return is symmetric, the investor's utility function is quadratic or of lower order, or if the mean and the variance are sufficient to define the distribution of returns, as in the case of the normal.

Empirically, skewness has received considerable attention in the field of asset pricing, where models have been developed to incorporate a preference for positively skewed assets within the Capital Asset Pricing Model (CAPM). Kraus and Litzenberger (1976) developed and tested a three-moment extension of the Sharpe-Lintner CAPM, finding evidence that systematic skewness is priced in the market, and that investors have a preference for positive skewness. Their paper is criticized by Friend and Westerfield (1980), who provide contrary evidence on the explanatory power of the three-moment CAPM, but yet agree on the fact that investors prefer positive skewness and that they are willing to pay a premium to be able to hold positively skewed assets. Further contribution is brought by Harvey and Siddique (2000), who study an asset pricing model incorporating conditional skewness and coskewness, with analogous results about investors preferences. Barberis and Huang (2008) put Cumulative Prospect Theory, developed in Tversky and Kahneman (1992), in an asset pricing perspective, showing that skewness of individual securities is priced, not just their systematic skewness, and that such feature is highly desirable for investor, who in turn are willing to pay a premium for positively skewed assets. Further, they provide an explanation to the observed phenomenon of under-diversification in investors' individual portfolios, motivated by individuals' appetite for skewed distributions of their portfolio returns.

The above is in line with Mitton and Vorkink (2007), who find that investors are willing to trade off the benefits of a diversified portfolio to achieve a higher level of portfolio positive skewness, implying that investors with a preference for positive skewness would deliberately hold a suboptimal portfolio from a mean-variance perspective as described by Markowitz

(1952), Sharpe (1966), according to which the set of efficient portfolios will always exhibit the highest attainable Sharpe-ratio, if a riskless asset is available on the market, and by Tobin (1958) famed two-fund separation theorem. An investor with strong preference for positive skewness, might eventually deliberately choose to hold a portfolio that exhibits a smaller Sharpe-ratio than the best attainable one among the available set of assets. Further critique to the mean-variance approach can be found on Simkowitz and Beedles (1978), who point at the trade-off between diversification, with low risk, and high positive portfolio skewness, and to how the mean-variance criterion fails to consider its implications. Samuelson (1970) provides a defence and a critique of the mean-variance approach at the same time, as he proves that the mean-variance becomes an adequate method when portfolio decisions are made continuously, but questions its adequacy in cases when portfolio decisions are made less frequently.

Different attempts have been made to address the issue of portfolio choice with skewness. Chunhachinda et al. (1997) incorporate investor's preference for skewness within a technique named Polynomial Goal Programming, to find the investor's optimal portfolio between 14 international stock market indices. Their results are in line with previous theory, finding that investors with a high preference for skewness will hold relatively under-diversified portfolios to attain a higher level of positive portfolio skewness, and that the same investor will hold a considerably different portfolio when the preference for skewness is introduced, as opposed to when it is not. Unfortunately, the PGP is not connected to a utility function, or the expectation of it. De Athayde and Flores (2004) generalize the three-moment allocation problem by introducing a methodology to find the mean-variance-skewness efficient frontier trying to minimize portfolio variance for given values of portfolio return and portfolio skewness, analogously to the well-known mean-variance efficient frontier to illustrate the set of optimal portfolios an investor can choose from, according to her personal preferences. Intuitively, while the set of portfolios belonging to the mean variance frontier can be represented graphically by a line in a mean-variance space, the three dimensional mean-variance-skewness frontier will be represented by a surface in a three dimensional plane. While these approaches address the issue of portfolio allocation with skewness, none of the two deals with an accurate description of an investor set of preference embedded in a utility function of wealth, or its approximation via a Taylor expansion. The issue is addressed under this light by Harvey et al. (2010) and Jondeau and Rockinger (2006). The former addresses both the issue of portfolio selection with higher moments than mean and variance, together with the one of estimation error, by means of Bayesian techniques. The latter is the one that more closely relates to our research, as our methodologies are similar. Jondeau and Rockinger (2006) use of the truncated Taylor series expansion to approximate a utility function of wealth exhibiting the property of Constant Relative Risk Aversion (CARA) up to the second, third and fourth moment, to study how a portfolio composition changes with the different degree of approximation and various levels of risk aversion. They compare the different portfolio optimizations with a “direct optimization” technique developed by Simaan (1993) and they see how worse off an investor would be in terms of utility when the allocation is obtained by maximizing the Taylor approximation of the utility function of wealth, truncated at different points to include or exclude preferences for higher moments, compared to direct optimization. One of their conclusions is that, when the departure from normality in the multivariate distribution of the assets under consideration is particularly relevant, neglecting skewness from allocation criteria will lead to a very different portfolio composition, leaving the investor worse off, with a loss of expected utility. While Jondeau and Rockinger (2006) provide good intuition in the field of portfolio choice with utility maximization, their approach is limited by their use of the exponential utility function alone. A utility function exhibiting CARA, while very popular in the literature due to its mathematical tractability, fails to realistically and comprehensively describe investor preferences, and therefore limits the extent of their analysis. They chose to highlight how a mean-variance-skewness approximation becomes increasingly relevant as skewness becomes more pronounced in the underlying multivariate distribution of return for the set of assets under consideration. That is, the more skewed the underlying assets’ returns are, the higher the difference between optimal portfolios when a mean-variance-skewness approximation of utility is in place, as compared to a mean-variance one.

We are different in our approach, as we generalize the problem to study how the investor’s choice changes when different kinds of risk aversion are in place, rather than looking at the problem by using a CARA utility function alone. Additionally, we introduce a risk-free rate in the investment problem. To quantify differences, we will additionally make use of the certainty equivalent as a comparison tool, to quantify the differences in expected utility between MV and MVS investors in the different cases with a graspable economic measure.

This calls the need of a common framework that would allow us to compare between different kinds of preferences. We chose to develop such framework, by formulating a generalized utility function of wealth that can be easily adjusted in its parameters to fit different types of relative and absolute risk aversions. The ground for this approach is provided by Arrow (1965) and Pratt (1964), which developed independently the concepts of absolute and relative risk aversion (ARA and RRA) to describe the utility function, as well as quantitative measures for it, defining what are known as the Arrow-Pratt functions of absolute (A(W)) and relative (R(W)) risk aversion. Pratt (1964) also describes the constant relative risk aversion (CRRA) utility function with power utility, as well as the constant absolute risk aversion (CARA) utility function with exponential utility. Both of these functions find broad employment within financial literature. CARA implies that an investor would invest the same dollar amount into risky assets even as her wealth increases or decreases. This is in contrast to CRRA which implies that an investor would invest the same percentage out of wealth into risky assets for different level of wealth. Arrow (1965) further shows that a utility function is defined up to a positive affine transformation, meaning that the utility function remains the same if you multiply it by a positive constant or by adding some constant to it. It is also shown that the absolute risk aversion completely characterizes the utility function. Further, the concept of variable risk aversion is introduced, covering the spectrum from CARA to CRRA, when the utility function exhibits DARA and IRRA simultaneously. Kane (1982) extends the concepts of risk aversion, by presenting a measure for relative skewness preference, similar to RRA, as well as a skewness ratio, i.e. relative skewness preference over relative risk aversion (and applied it to power and exponential utility). Further, it is shown that there is no feasible solution to the allocation problem when skewness is too high. It is also mentioned that decreasing absolute risk aversion (DARA) is widely accepted but that the need for increasing relative risk aversion (IRRA) is debatable.

As an example of generalized utility function in the literature, Merton (1971) makes uses the hyperbolic absolute risk aversion (HARA), a ductile utility function suited to model different kinds of risk aversion, made easily accessible by an adjustment of its parameters. The HARA utility function is a powerful tool, as it is very general and can include CARA and CRRA as well as other kinds of risk aversions, but presents however some important limitations that prevents us to use it directly in our analysis, as it cannot cover DRRA without violating Scott and Horvath (1980) criteria for a risk averse investor with strict consistent preference for moments at all wealth levels. We will discuss such limitations more in detail in the following section.

Finally, for the sake of completeness, let us outline a potential shortcoming of the approximation of utility by using a Taylor series expansion. Hassett et al. (1985) argue that the Taylor expansion does not always converge to the actual utility function, for example in the case of very large (or very negative) returns. They further show that truncations at mean-variance or mean-variance-skewness might provide a poor approximation of the utility function, especially with highly skewed assets, which is in line with what is predicted by Kane (1982). Our analysis is however not shaken by such concerns. The values for skewness and returns that are used in our empirical analysis are far from those of the options used in that of Hassett et al. (1985) to argue on the limits of the Taylor approximation. Options can display extreme values of return and skewness: one example they use is that returns above 140% does not work in the Taylor series. Numbers of this magnitude are not displayed by the return distributions we consider to illustrate our intuition in the empirical section of this paper, but one should bear such limitation in mind in case of further experimentation with different sets of assets.

4 Theoretical framework

In the theoretical framework we will lay the foundation for our empirical studies. We will start with discussion of the investment decision that an investor faces and continue with the utility function we use to solve it.

4.1 The investment problem

To start our analysis, we go through the traditional investment problem with our assumptions, what the investor cares about and how one can solve the problem with a Taylor series expansion of the expected utility.

4.1.1 Assumptions

We consider a single-period investment problem for a utility maximizing individual. Assume a rational investor who wishes to invest her initial wealth W_0 at the beginning of the period, in a way that maximizes her utility $U(W)$ for the end-of-period wealth, W. The investor can invest her wealth across N risky assets, each with return r_i, collectively summarized by return vector $r = (r_1, \ldots, r_n)'$ with joint cumulative distribution function $F(r)$. The investor allocates a fraction of wealth θ_i into the i-th asset summarized in vector notation by the set of portfolio weights $\theta = (\theta_1, \dots, \theta_n)'$. There could be a risk-free asset and in such case r_f will be added to the return vector and θ_f to the portfolio weights vector. Her portfolio will therefore have stochastic portfolio return $r_p = \theta' r$ and the end-of-period wealth will then be $W = W_0 (1 + r_p)$. Whether there is a riskless asset available or not, the weights need to sum to one for initial wealth W_0 to be fully invested in the risky assets and the potential risk-free asset. Further, short-selling is not made available, hence no weight in any asset can be negative. The problem can therefore be formalized as:

$$\max_{\theta} U(W)$$

$$s.t \sum_{i=1}^{N} \theta = 1, \quad \theta_i \geq 0, \forall\, i$$

4.1.2 Expected utility

The investor thus faces a choice under uncertainty, as end-of-period return is unknown at the moment of investment. The choice then becomes a maximization of the expected utility of end-of-period wealth $E[U(W)]$. The expected utility can therefore be defined as:

$$E[U(W)] = \int U(W) f(W) dW$$

where $f(W)$ is the probability distribution function of end-of-period wealth. The distribution $f(W)$ depends on the underlying multivariate distribution of returns, as well as the set portfolio weights θ. Hence, to solve this problem one would need to know the joint cumulative distribution function of assets returns or have an empirical joint cumulative distribution function of assets returns. As expressed in Jondeau and Rockinger (2006), the problem is easily solved with an empirical distribution, but generally does not have a closed-form solution when dealing with a parametric joint distribution, and might become computationally expensive.

4.1.3 Taylor expansion

A feasible and popular way to overcome this issue is to make an approximation of utility by using a truncated Taylor series expansion. The infinite-order Taylor series expansion of a utility function around expected end-of-period wealth, as introduced by Hassett et al. (1985), is defined as:

$$U(W) = \sum_{m=0}^{\infty} \frac{U^{(m)}(\bar{W})\,(W - \bar{W})^m}{m!}$$

$$\bar{W} = E[W] = W_0\left(1 + \mu_p\right) = W_0(1 + \theta'\mu), \qquad \mu = E[r]$$

Taking expectations on both sides gives:

$$E[U(W)] = E\left[\sum_{m=0}^{\infty} \frac{U^{(m)}(\bar{W})\ (W - \bar{W})^m}{m!}\right] = \sum_{m=0}^{\infty} \frac{U^{(m)}(\bar{W}) E[\ (W - \bar{W})^m]}{m!}$$

This means that the expected utility can be expressed in terms of central moments of the end-of-period wealth probability distribution and the derivatives of the utility function. Hence, by arbitrarily truncating the expansion at different values of m, we can choose whether to include moments of increasingly higher order in the equation. A mean-variance investor, for instance, would disregard higher moments above variance, hence the Taylor approximation of her utility function will be truncated at $m = 2$. Conversely, a mean-variance-skewness investor will exhibit a Taylor approximation truncated at $m = 3$, displayed below:

$$E[U(W)] \approx U(\bar{W}) + U'(\bar{W})E[W - \bar{W}] + \frac{U''(\bar{W})}{2}E[(W - \bar{W})^2] + \frac{U'''(\bar{W})}{6}E[(W - \bar{W})^3]$$

4.1.4 Moments and central moments

To better understand the benefits of using the Taylor series expansion to approximate utility, let us first take a step back and briefly explain the concept of moments and central moments.

In statistics, a moment is a numerical measure that can be used to describe the shape of a probability distribution. The mean is the first moment of a distribution, and provides information about its central tendency.

Central moments are a subset of statistical moments, defined around the mean. They are the expected value of the deviation of a variable from the mean, to the power of a specified integer. As an example, take end-of-period wealth W as our variable under consideration and its mean $\bar{W}$. If we set the value of the specified integer to be equal to 2, we have the variance, if we set it to 3, we have the skewness.

$$E[(W - \bar{W})^2] = \sigma_W^2$$

$$E[(W - \bar{W})^3] = s_W^3$$

For the sake of completeness, we can also see that the first central moment equals zero.

$$E[W - \bar{W}] = E[W] - \bar{W} = 0$$

Worth mentioning is that while variance is the same as used in ordinary statistics, the measure for skewness we use throughout our analysis is different from the common standardized one. While standardized skewness is defined as s^3/σ^3 . Whenever we refer to skewness in the paper, we refer to s^3 (or just s, in the tables and graphs of results).

4.1.5 Taylor approximation with portfolio moments

Now that moments and central moments are clear, let us see how they relate to our portfolio choice problem.

While the first central moment had been shown to be zero, the second and the third central moments of wealth can be simplified as the central moments of the distribution of portfolio returns:

$$E[(W-\bar{W})^2] = W_0^2\, E\left[\left(\left(1+r_p\right)-\left(1+\mu_p\right)\right)^2\right] = W_0^2\, E\left[\left(r_p-\mu_p\right)^2\right] = W_0^2\sigma_p^2$$

$$E[(W-\bar{W})^3] = W_0^3\, E\left[\left(r_p-\mu_p\right)^3\right] = W_0^3 s_p^3$$

Where σ_p^2 is the variance of portfolio returns and s_p^3 is central skewness of the portfolio. This allows us to rewrite the approximation as:

$$E[U(w)] \approx U(\bar{w}) + \frac{w_0^2 U''(\bar{w})}{2}\sigma_p^2 + \frac{w_0^3 U'''(\bar{w})}{6} s_p^3$$

To be able to calculate portfolio variance and portfolio skewness, one needs to know the covariance and coskewness structures of the underlying multivariate return distribution. Such structure is provided for by the covariance matrix Σ, defined as:

$$\Sigma = M_2 = E[(r-\mu)(r-\mu)'] = \{\sigma_{ij}\}$$

$$\sigma_{ij} = E\left[(r_i-\mu_i)\left(r_j-\mu_j\right)\right] \quad i,j = 1,\dots,N$$

And by the coskewness matrix, S, defined as:

$$S = M_3 = E[(r-\mu)(r-\mu)'\otimes(r-\mu)'] = \{s_{ijk}\}$$

$$where \otimes is\ the\ Kronecker\ product$$

$$s_{ijk} = E\left[(r_i-\mu_i)\left(r_j-\mu_j\right)(r_k-\mu_k)\right] \quad i,j,k = 1,\dots,N$$

The covariance matrix provides information about the dispersion of the assets' returns around their mean, as well as their tendency to change together. In an analogous way, the coskewness matrix provides information about the skewness of the individual assets' return, as well as their tendency to assume extreme values together. N being the number of assets under consideration, the coskewness matrix has dimensions $N \times N$, while the coskewness matrix will have dimension $N \times N^2$. Further, the covariance matrix is symmetrical and the coskewness is made up by N symmetrical matrices. When a risk-free asset is introduced the dimensions of the matrices will proportionally increase just as if there were $N+1$ assets. The new cells will contain only zeros since a theoretical risk-free rate have no variance, no covariance with other assets, no skewness and no co-skewness with the risky assets.

Portfolio variance and the portfolio skewness can now be computed as:

$$\sigma_p^2 = \theta' M_2 \theta = E\left[\left(r_p - \mu_p\right)^2\right]$$

$$s_p^3 = \theta' M_3 (\theta \otimes \theta) = E\left[\left(r_p - \mu_p\right)^3\right]$$

$$where \otimes is the Kronecker product$$

4.1.6 Certainty equivalent

Additionally, we introduce the certainty equivalent, a standard concept of utility theory and applied by several authors. It will be an important tool to our empirical analysis.

For a utility maximizing investor, the certainty equivalent is the amount of certain return that would yield the same level of expected utility of the uncertain return of the risky portfolio. It is defined as:

$$U(1 + CE) = E\left[U\left(1 + r_p\right)\right]$$

$$CE = U^{-1}\left(E\left[U\left(1 + r_p\right)\right]\right) - 1$$

The certainty equivalent allows to economically quantify the level of expected utility of wealth of a given portfolio, as it is expressed in the same quantity as the return of the portfolio itself. When the investor tries to maximize expected utility, she is trying to maximize the certainty equivalent.

Now the portfolio choice problem is clear. The investor needs to choose the appropriate combination of portfolio weights, in order to obtain the desired mixture of portfolio return, variance and skewness according to her preferences, specified in the utility function. We should then discuss the investor’s utility function.

4.2 The utility function

In financial economics, the utility function provides information about an investor’s set of preferences in the trade-off between risk and return. Different aversion to risk can be captured by different utility functions.

4.2.1 Risk Aversion

Let us briefly review the different kinds of risk aversion first. Constant absolute risk aversion (CARA), which finds broad employment in financial literature, implies that an investor would invest the same dollar amount into risky assets for different levels of wealth. Constant relative risk aversion (CRRA) on the other hand, which also finds a variety of applications in the literature, implies that an investor invests the same proportion of wealth into risky assets, irrespective of the level of wealth. Extending absolute risk aversion, there is also increasing

ARA (IARA) which implies that an investor would invest less (in dollar amount) into risky assets when her wealth increases. In the other direction there is decreasing ARA (DARA) which analogously implies that an investor will invest more dollars in risky assets since she becomes less risk averse with wealth. DARA covers the region in between CARA and CRRA, and can further extend beyond CRRA.

It is not just DARA that covers the area in between CARA and CRRA but also increasing RRA (IRRA) which implies that the proportion of wealth invested into risky assets decreases as wealth increases. Thus, IRRA includes IARA, CARA and a part of DARA. On the other side of CARA is decreasing RRA (DRRA) which, analogously, implies that the proportion invested into risky assets increases with wealth.

The Arrow-Pratt absolute and relative risk aversions functions are defined as

$$A(W) \equiv -\frac{U''(W)}{U'(W)}, \quad R(W) \equiv A(W)W = -\frac{WU''(W)}{U'(W)}$$

And we find what type of risk aversion the investor has by taking the derivative of the risk aversion functions

$$\frac{dA(W)}{dW}\begin{cases} > 0 & IARA \\ = 0 & CARA \\ < 0 & DARA \end{cases}, \quad \frac{dR(W)}{dW}\begin{cases} > 0 & IRRA \\ = 0 & CRRA \\ < 0 & DRRA \end{cases}$$

As we are interested in learning more about how the skewness dimension impacts the portfolio choice problem through the utility function, we will also define two other measures introduced by Kane (1982): relative skewness preference $S(W)$, notably similar to relative risk aversion, and skewness ratio $K(W)$ which is relative skewness preference over relative risk aversion. They are defined as:

$$S(W) \equiv W^2 \frac{U'''(W)}{U'(W)}, \quad K(W) \equiv -\frac{WU'''(W)}{U''(W)} = \frac{S(W)}{R(W)}$$

They will be an important tool for our analysis to understand how skewness preference increases or decreases in relevance for different investors.

4.2.2 Limitations of HARA

In order for us to have a comparable ground and look for portfolio similarities in a mean-variance case compared to a mean-variance skewness case, we need to have a generalized utility function that could capture the whole spectrum of risk aversions with different parameters, and which would eventually allow for comparable results across different investors. Therefore, we need a utility function that can fit the full set of cases from CARA to

DRRA. Several financial authors have suggested the use of the HARA family of utility functions, but because of the mathematical tractability of its special cases CARA and CRRA most have used only these. This has created a gap in the literature using something in between as well as outside these special cases.

The HARA utility, as expressed in Merton (1971) is

$$U_{HARA}(W) = \frac{1-\gamma}{\gamma}\left(\frac{\beta W}{1-\gamma} + \eta\right)^{\gamma}$$

The utility function can exhibit DRRA when $\eta < 0$ $(-\infty < \gamma < 1)$ but then the first derivative

$$U'_{HARA}(W) = \beta\left(\frac{\beta W}{1-\gamma} + \eta\right)^{\gamma-1}$$

will become infinite when the wealth W is such that the term within the parenthesis equal zero. When the term inside the parenthesis is negative you get either negative or complex values depending on whether gamma is an integer or not. According to Scott and Horvath (1980), for a risk averse investor with strictly consistent preferences for moments, odd derivatives need to be positive, while even derivatives need to be negative, for all wealth levels. Thus HARA utility does not have a DRRA alternative that can be used in portfolio allocation since in the described cases the derivatives do not fulfil these criteria.

As the existing literature does not seem to provide a ready-to-use utility function that might be suitable to our purposes, we have chosen to tackle the problem by deriving our own. Since there are different utility functions that cover various types of risk aversion, but no one that covers them all, we merged two different existing utility functions in order to derive a generalized one, that could fit our purposes. By doing so, we would be able to cover the full spectrum of risk aversions that is necessary for our analysis.

4.2.3 A utility function exhibiting DRRA

The first utility function we take into consideration is a Bernoulli function of the generalized form:

$$U(W) = -e^{\beta W^{-\gamma}}$$

This function is very interesting because it can exhibit DRRA without limitation, which was one of the shortcomings of HARA functions explained above. Let us look at it in more detail.

The functions first and second derivative are:

$$U'(W) = \beta\gamma W^{-(\gamma+1)}e^{\beta W^{-\gamma}}$$

$$U''(W) = -\beta\gamma W^{-2(\gamma+1)} e^{\beta W^{-\gamma}} (\beta\gamma + (\gamma+1)W^{\gamma})$$

Since the first derivative should be positive, we find that β and γ must have the same sign and that none of them can be zero. From the second derivative, that needs to be negative for a risk averse investor, we find that $\gamma \geq -1$. Higher derivatives do not give further limitations in parameters. Let us continue by looking at the Arrow-Pratt measures of risk-aversion:

$$A(W) \equiv -\frac{U''(W)}{U'(W)} = W^{-(\gamma+1)}(\beta\gamma + (\gamma+1)W^{\gamma})$$

$$R(W) \equiv A(W)W = W^{-\gamma}(\beta\gamma + (\gamma+1)W^{\gamma}) = \beta\gamma W^{-\gamma} + \gamma + 1$$

Taking the derivative of the $A(W)$ and $R(W)$ and bearing in mind the parameter limitations described above, we can find when the function exhibits different types of risk aversion.

$$\frac{dA(W)}{dW} = -W^{-(\gamma+2)}(\gamma+1)(\beta\gamma + W^{\gamma}) \begin{cases} > 0 & \textit{not possible} & IARA \\ = 0 & \gamma = -1 \to \beta < 0 & CARA \\ < 0 & \gamma > -1 & DARA \end{cases}$$

$$\frac{dR(W)}{dW} = -\beta\gamma^2 W^{-(\gamma+1)} \begin{cases} > 0 & \beta < 0 \to -1 \leq \gamma \leq 0 & IRRA \\ = 0 & \textit{not possible} & CRRA \\ < 0 & \beta > 0 \to \gamma > 0 & DRRA \end{cases}$$

We see that CRRA is not possible since it would suggest beta or gamma equals zero which is a violation of Scott and Horvath (1980) criteria, as would further yield a constant utility function. Thus, this function cannot exhibit CRRA. We therefore need to find a way to extend the function to incorporate this shortcoming.

4.2.4 A utility function with variable risk aversion

The second utility function under observation exhibits the property of variable risk aversion, meaning that we can therefore work with its parameters to obtain a function exhibiting CARA, CRRA, or something in between the two extremes, hence exhibiting different degrees of DARA and IRRA. It is defined by its relative risk aversion (recall that this is sufficient) as

$$R(W) \equiv A(W)W = \alpha + \beta W \to A(W) = \frac{\alpha}{W} + \beta$$

For investor to be risk averse for all wealth levels, either alpha or beta needs to be positive or zero, but both cannot be simultaneously zero. Once again we can investigate the different types of risk aversion by taking the derivatives of $A(w)$ and $R(w)$.

$$\frac{dA(W)}{dW} = -\frac{\alpha}{W^2} \begin{cases} > 0 & \textit{not possible} & IARA \\ = 0 & \alpha = 0 & CARA \\ < 0 & \alpha > 0 & DARA \end{cases}$$

$$\frac{dR(W)}{dW} = \beta \begin{cases} > 0 & \beta > 0 & IRRA \\ = 0 & \beta = 0 & CRRA \\ < 0 & not\ possible & DRRA \end{cases}$$

We thus find that this utility function does not exhibit DRRA, but we confirm that it is capable of exhibiting CRRA.

4.2.5 The generalized utility function

By taking a look at the derivative of the last utility function under consideration (which you find by solving the differential equation above) we have:

$$U'(W) = W^{-\alpha} e^{-\beta W}$$

We see that it is very similar to the first derivative of the first DRRA utility function under consideration, thus it might work to combine them. By combining these two derivatives we have:

$$U'(W) = W^{-\alpha} e^{-\beta W^{\gamma}}$$

This is a solution to the differential equation (solving for $U'(W)$)

$$R(W) \equiv A(W)W \equiv -\frac{U''(W)W}{U'(W)} = \alpha + \beta\gamma W^{\gamma}$$

Finally, taking the integrating this derivative yields our final utility function:

$$U(W) = \begin{cases} -\dfrac{W^{1-\alpha}\,(\beta\, W^{\gamma})^{\frac{\alpha-1}{\gamma}}\,\Gamma\left(\frac{1-\alpha}{\gamma}\middle|\beta\, W^{\gamma}\right)}{\gamma} & \beta \neq 0 \\ \dfrac{W^{1-\alpha}}{1-\alpha} & \beta = 0, \alpha \neq 1 \\ \log(W) & \beta = 0, \alpha = 1 \end{cases}$$

$$\text{where } \Gamma(a|x)\ is\ the\ incomplete\ gamma\ function$$

The above is a parametric utility function of wealth, that can take different shapes for different values of the three parameters α, $\beta$and γ.

4.2.6 Properties and characteristics of the generalized utility function

Let us explore its properties. The second to fourth derivatives are:

$$U''(W) = -W^{-\alpha-1} e^{-\beta W^{\gamma}} (\alpha + \beta\gamma W^{\gamma})$$

$$U'''(W) = W^{-\alpha-2} e^{-\beta W^{\gamma}} \big((\alpha + \beta\gamma W^{\gamma})^2 + \alpha + \beta\gamma W^{\gamma}(1-\gamma)\big)$$

$$U''''(W) = -W^{-\alpha-3} e^{-\beta W^{\gamma}} \big((\alpha + \beta\gamma W^{\gamma})^3 + 3\alpha^2 + 3(\beta\gamma W^{\gamma})^2(1-\gamma) + 3\alpha\beta\gamma W^{\gamma}(2-\gamma) + 2\alpha + \beta\gamma W^{\gamma}(2-\gamma)(1-\gamma)\big)$$

Again, every odd derivative needs to be positive for all wealth levels and every even derivative needs to be negative for all wealth levels. From how the derivatives are presented it can be seen that $\alpha \geq 0 \,\&\, \beta\gamma \geq 0 \,\&\, \gamma \leq 1 \,\&\, when\ \alpha = 0 \rightarrow \beta\gamma \neq 0$. Further, it can be seen that it would make no sense to use $\gamma = 0$ since this would just create an affine transformation of the utility function with $\beta = 0$.

Investigating further for these possible parameter values, we find that the utility function (when $\beta \neq 0$) can be rewritten as:

$$U(W) = -\frac{\beta^{\frac{\alpha-1}{\gamma}}\Gamma\left(\frac{1-\alpha}{\gamma}\middle|\beta\, W^{\gamma}\right)}{\gamma}$$

Also, since β can take negative values, numerical analysis further reveals that the utility function might take complex values. Even though this might seem as a problem, it is not. These complex values are constant for fixed parameters and can disappear by an affine transformation or adequately by adding an integration constant in the derivation of the utility function.

We should further examine the properties of our utility function using Arrow-Pratt's absolute and relative risk-aversion functions, $A(W)$ and $R(W)$, and Kane (1982) relative skewness preference $S(W)$, and skewness ratio, $K(W)$. We have

$$A(W) \equiv \frac{-U''(W)}{U'(W)} = \frac{\alpha + \beta\gamma W^{\gamma}}{W} = \frac{\alpha}{W} + \beta\gamma W^{\gamma-1}$$

$$R(W) \equiv WA(W) = \alpha + \beta\gamma W^{\gamma}$$

$$S(W) \equiv \frac{W^2 U'''(W)}{U'(W)} = (\alpha + \beta\gamma W^{\gamma})^2 + \alpha + \beta\gamma W^{\gamma}(1-\gamma)$$

$$K(W) \equiv \frac{S(W)}{R(W)} = \alpha + \beta\gamma W^{\gamma} - \frac{\beta\gamma^2 W^{\gamma}}{\alpha + \beta\gamma W^{\gamma}} + 1 = \alpha + \beta\gamma W^{\gamma} + \frac{\alpha\gamma}{\alpha + \beta\gamma W^{\gamma}} - \gamma + 1$$

By taking the derivative of $A(W)$ and $R(W)$ we can find which parameters corresponds to increasing, constant and decreasing absolute risk aversion (IARA, CARA, DARA) as well as increasing, constant and decreasing relative risk aversion (IRRA, CRRA, DRRA). If we include parameters limitations in the analysis, to comply with the criteria defined in Scott and Horvath (1980), we also find which risk aversion types that are possible.

$$\frac{dA(W)}{dW} = -\frac{\alpha}{W^2} - \beta\gamma W^{\gamma-2}(1-\gamma)\begin{cases} > 0 & not\ possible & IARA \\ = 0 & \alpha = 0 \,\&\, \gamma = 1 & CARA \\ < 0 & \alpha > 0 & DARA \end{cases}$$

$$\frac{dR(W)}{dW} = \beta\gamma^2 W^{\gamma-1}\begin{cases} > 0 & \beta > 0 \rightarrow \gamma > 0 & IRRA \\ = 0 & \beta = 0 & CRRA \\ < 0 & \beta < 0 \rightarrow \gamma < 0 & DRRA \end{cases}$$

For completeness, we should also show the derivative of $S(W)$

$$\frac{dS(W)}{dW} = \beta\gamma^2 W^{\gamma-1}(2\alpha + 2\beta\gamma W^{\gamma} - \gamma + 1)$$

Thus, the utility function can achieve all types of risk aversion that are considered realistic, including CARA and CRRA that have been heavily used so far in the literature. Our utility function is not universal, in a sense that the IARA case cannot be covered. It is straightforward to say however that IARA would not make any sense intuitively, as it implies that an investor would invest less wealth into risky assets in dollar amount as wealth increases. It does not seem likely, and thus we disregard this case from our analysis going forward. CARA also does not seem too realistic in real terms, although more plausible than the IARA case. We will hence look at CARA as one end of the spectrum of different types of risk aversion.

4.2.7 Parameters choice

To cover the different types of risk aversion one must choose the different parameters that will define our utility function in the different cases. We do so by looking at two measures, the first being the overall level of ARA and RRA captured by $A(W)$ and $R(W)$, and the sign of their first derivatives dA/dW and dR/dW. For the sake of simplicity, we will assume the case $W = 1$, so that $A(W) = R(W)$.

By setting an arbitrary value for $A(W)$ and $R(W)$ and the desired combination of signs for the two derivatives dA/dW and dR/dW we can work our way backwards, to find a combination of parameters that defines our utility function for the desired type of risk aversion with the desired level of risk aversion. Let us explain this more clearly with a numerical example. Consider an individual with overall level of risk aversion $A = R = 5$ and DARA utility ($dA/dW = -1$). We can find out which values of α, $\beta$and γ are needed to define its utility function by solving the system of equations

$$\begin{cases} A(W)|_{W=1} = R(W)|_{W=1} = \alpha + \beta\gamma = 5 \\ \left.\dfrac{dA(W)}{dW}\right|_{W=1} = -\alpha - \beta\gamma(1-\gamma) = -1 \end{cases}$$

For α, $\beta$and γ. This would yield endless opportunities of parameters since we have two equations and three unknown, but if one includes the derivative of skewness preference a unique solution emerges. The explained methodology can be used both to define utility functions with different types of absolute and relative risk aversions with the same level of overall risk aversion, as well as utility function with different levels of risk aversion for the same type of absolute and relative risk aversion.

4.2.8 Further comments on parameters selection

The choice of including three discrete parameters in our utility function allows us to control what kind of risk aversion (from CARA to DRRA) an investor will exhibit for given levels of risk aversion. One could argue that it should be possible to change the parameterization into just two parameters, which should be sufficient to define the whole spectrum of risk aversions from CARA to DRRA, and thus take away unnecessary complexity from the new utility function. The full set of combinations of the three parameters however allow us to have complete control not only on the first derivative of $A(W)$ and $R(W)$, defining the types of ARA and RRA, but also on the second derivative of $A(W)$ and $R(W)$ or adequately the derivative of $S(W)$. This means that, by adjusting the three parameters altogether, we could not only choose what kind of risk aversion the investor has, but also its slope. Let us say we want an investor in the DRRA case. We would then adjust the parameters to have the first derivative of $A(W)$ and $R(W) < 0$. By playing with the parameter further though, we could also have control on the slope of those derivative, in this case defining how quickly the investor's risk aversion would decrease as wealth increases.

4.2.9 Utility function in portfolio allocation

We have so far found a way to optimize the portfolio allocation by the use of a Taylor expansion around expected end-of-period wealth. Also, we have developed a utility function to be used in this Taylor expansion. Let us now put everything together and express the Taylor expansion with our utility function:

$$E[U(W)] \approx U(\bar{W}) + \frac{W_0^2 U''(\bar{W})}{2}\sigma_p^2 + \frac{W_0^3 U'''(\bar{W})}{6} s_p^3 =$$

$$-\frac{\beta^{\frac{\alpha-1}{\gamma}}\Gamma\left(\frac{1-\alpha}{\gamma}\middle|\beta\,\bar{W}^{\gamma}\right)}{\gamma} - W_0^2 \bar{W}^{-\alpha-1} e^{-\beta\bar{W}^{\gamma}}(\alpha + \beta\gamma\bar{W}^{\gamma})\frac{\sigma_p^2}{2}$$

$$+ W_0^3 \bar{W}^{-\alpha-2} e^{-\beta\bar{W}^{\gamma}}\left((\alpha + \beta\gamma\bar{W}^{\gamma})^2 + \alpha + \beta\gamma\bar{W}^{\gamma}(1-\gamma)\right)\frac{s_p^3}{6}$$

$$when\ \beta \neq 0$$

It would reduce the complexity if we, as many else, could simplify the initial wealth to one without loss of generality. We try to do this with $\beta = b/W_0^{\gamma}$ and find that the result is an affine transformation of the expected utility with a constant based on initial wealth and α.

This allows us to simplify initial wealth as equal to one without loss of generality. If someone would like to use a different level of initial wealth, it would be sufficient to adjust the beta parameter to find consistent results from an optimization or calculation.

5 Empirical Analysis

In the previous section, we have extensively explained the theory at the basis of our analysis. We will now advocate for the importance of skewness in asset allocation by doing a series of portfolio optimization for different investors, using empirical data.

5.1 Setup for the optimization

5.1.1 Data selection and input derivation

We will now implement a series of sets of portfolio optimizations to analyse the choices of a diverse group of investors across a variety of assets. For each set of asset, we will look at different samples of historical prices to calculate the time series of their monthly returns which will then be used to derive the inputs to the portfolio optimization. As we are approximating the expected utility with a Taylor series expansion using the moments of the return distribution, in order for us to do the portfolio optimization it is sufficient to know the vector of expected returns $E[r]$, the variance covariance matrix Σ and the coskewness matrix $M_3\ or\ S$.

We approximate the inputs from a sample of historical return of the assets under consideration. Note that portfolios that have been optimized with inputs derived using such method have been widely proven not to perform well out-of-sample. The aim of this thesis, however, is to provide a comparison between MV and MVS optimized portfolios in a static setting, while the out-of-sample performance of the portfolio is beyond the scope of our research and, as that, it would add unnecessary complexity to our analysis.

We consider three broad sets of assets in four data sets, to ensure that our results are consistent across the analysis, and for each data set we optimize our portfolio with and without considering a riskless asset.

The first data set includes 20 MSCI country indexes of developed markets, observed monthly between October 2011 to August 2016. MSCI indexes track the return of broadly diversified portfolios designed to capture 85% of the total market capitalization in a given region. As mentioned by Jondeau and Rockinger (2006), we expect such indexes to be well diversified exhibiting a lower level of either positive or negative skewness and to behave more closely as a normal distribution. Further, as previous theory suggests, preference for positive skewness might encourage investors to hold less diversified portfolio to achieve a greater level of positive skewness. Hence having only well diversified portfolios as the only available assets might not give an opportunity to the investor to trade off a meaningful level of diversification with a riskier but more positively skewed portfolio, as even investing her entire wealth in one asset would still mean to hold a diversified fund.

Hence, to see how investors would make different choices in cases where skewness is prevalent, we turn our attention to stocks. The first group of stocks under analysis are the constituents of the Swedish OMX30, the first 30 Swedish stocks by market capitalization. The choice of using Swedish large capitalization stocks is rather arbitrary, due to the larger availability of data and our desire to investigate the outcome of the portfolio choice problem using assets whose return exhibit a more idiosyncratic behaviour compared to well-diversified portfolios, as in the MSCI case. The choice of using a sample of 30 assets allows investors to reach a meaningful level of diversification and keeps the physical running time of our numerical optimization low. We derive the inputs by observing two different datasets of the same assets, the first one being made of monthly observations of stock prices between October 2011 and November 2016, the second made of monthly observation of stock prices between October 2003 and November 2016, to include stock prices behaviours during the 2007-08 financial crisis in the input derivation[1]. Finally, to ensure that our findings are consistent, we also analyse portfolios optimized using a sample of monthly returns of large cap German stock prices, the 30 constituents of the German index DAX30, observed between October 2011 and November 2016.

The same optimizations and subsequent analysis can be repeated by others, optimizing portfolios that take into consideration any desired set of assets, or input derivation methodology. For the purpose of this thesis, we felt that easy attainable data and an easy way to derive inputs were the most trivial way to analyse a possible empirical implication of our theoretical framework.

[1] The longer time series is excluding Nokia, since it was marketed on OMX Stockholm some years into the sample. We do not have any concerns that this would affect results.

5.1.2 Data description

We will now have a closer look to the sets of assets under consideration, by presenting some summary statistics and testing each asset's sample distribution for normality using the popular Wilk-Shapiro test at the 5% level, analogously to Chunhachinda et al. (1997). The Wilk-Shapiro determines the W-statistics, which tests the hypothesis that a given distribution is normal. If the p-value associated with the statistics is below a given level, 0.05 in our case, the hypothesis of normality is rejected at the 5% level.

5.1.2.1 MSCI 2011-2016

Table 1 – Summary statistics and normality testing for MSCI 2011-16 monthly returns

MSCI Index	Expected Return	Standard Deviation	Skewness	W-test	p-value	H_0
Australia	0.0531	0.1289	-0.0649	0.9607	0.0578	Not-Rejected
Austria	0.0268	0.1949	-0.0424	0.9739	0.2089	Not-Rejected
Belgium	0.1814	0.1436	0.0555	0.9875	0.7239	Not-Rejected
Canada	0.0407	0.0865	-0.0436	0.9788	0.4038	Not-Rejected
Denmark	0.1977	0.1540	0.0757	0.9682	0.1322	Not-Rejected
Finland	0.0959	0.1681	-0.0626	0.9795	0.3650	Not-Rejected
France	0.0786	0.1372	-0.0423	0.9893	0.8891	Not-Rejected
Germany	0.0998	0.1586	-0.0476	0.9900	0.9144	Not-Rejected
Hong Kong	0.0955	0.2018	-0.0813	0.9809	0.4148	Not-Rejected
Italy	0.0209	0.1961	-0.0876	0.9839	0.6358	Not-Rejected
Japan	0.1294	0.1898	-0.0644	0.9829	0.5841	Not-Rejected
Netherlands	0.1233	0.1353	-0.0635	0.9766	0.3217	Not-Rejected
New Zealand	0.1151	0.1218	0.0421	0.9839	0.5444	Not-Rejected
Norway	0.0235	0.1325	-0.0718	0.9599	0.0515	Not-Rejected
Singapore	-0.0021	0.1368	-0.0352	0.9851	0.6003	Not-Rejected
Spain	0.0224	0.2053	0.0719	0.9797	0.3718	Not-Rejected
Sweden	0.0860	0.1310	-0.0564	0.9784	0.3883	Not-Rejected
Switzerland	0.0824	0.1128	-0.0394	0.9815	0.4399	Not-Rejected
UK	0.0436	0.1059	-0.0526	0.9748	0.2287	Not-Rejected
US	0.1191	0.1047	-0.0448	0.9872	0.7095	Not-Rejected
Mean	0.0816	0.1473	-0.0328			
Standard Deviation	0.0526	0.0344	0.0493			

Table 1 provides annualized summary statistics for 20 MSCI indexes return time series, from a sample of monthly observations observed between October 2011 and August 2016, as well as the Wilk-Shapiro test statistic, each corresponding p-value, and the final decision on the null hypothesis for normality at the 5% level. The hypothesis of normality is rejected for level of p-value < 0.05

The table above summarizes the summary statistics for a sample of monthly return from October 2011 to August 2016 for 20 MSCI country indexes. It is immediately evident that none of the p-values in the table is lower than our confidence level of 0.05, hence the assumption of normality cannot be rejected for any asset. Because of that, we do not expect to see large differences between MV and MVS optimized portfolios using these assets, as skewness does not seem to be very prominent among these diversified portfolios.

We now turn our attention to individual stocks, an asset class that we expect to exhibit a more interesting behaviour and provide us with a more interesting set of inputs for our portfolio optimizations.

5.1.2.2 OMX30 2011-2016

Table 2 – Summary statistics and normality testing for OMX 2011-16 monthly returns

OMX30	Expected Return	Standard Deviation	Skewness	W-test	p-value	H_0
ABB	0.0959	0.1715	0.0558	0.9852	0.6791	Not-Rejected
Alfa Laval	0.0306	0.1939	0.0786	0.9812	0.4809	Not-Rejected
Assa Abloy	0.2456	0.1886	-0.0578	0.9775	0.3329	Not-Rejected
AstraZeneca	0.1202	0.2092	0.1209	0.9569	0.0344	Rejected
Atlas Copco (Class A)	0.1432	0.1942	-0.1050	0.9528	0.0212	Rejected
Atlas Copco (Class B)	0.1420	0.1902	-0.0940	0.9587	0.0409	Rejected
Electrolux	0.1500	0.2760	0.1215	0.9668	0.0932	Not-Rejected
Ericsson	-0.0535	0.2548	-0.1875	0.8987	0.0003	Rejected
Investor	0.2002	0.1747	-0.0820	0.9690	0.1308	Not-Rejected
Hennes & Mauritz	0.0496	0.1879	0.1014	0.9686	0.1248	Not-Rejected
Getinge	0.0044	0.2507	-0.0822	0.9557	0.0305	Rejected
Boliden	0.2095	0.3122	0.1530	0.9625	0.0596	Not-Rejected
Kinnevik	0.1360	0.2595	-0.0883	0.9844	0.6400	Not-Rejected
Lundin Petroleum	0.0373	0.2645	0.0881	0.9752	0.2605	Not-Rejected
Modern Times Group	-0.0261	0.2908	-0.1735	0.9590	0.0420	Rejected
Nokia	0.0812	0.4640	0.3463	0.8505	0.0000	Rejected
Nordea	0.1175	0.2223	0.0469	0.9869	0.7656	Not-Rejected
Sandvik	0.0579	0.2536	-0.0701	0.9846	0.5599	Not-Rejected
SCA	0.2164	0.1912	0.1181	0.9516	0.0204	Rejected
SEB	0.1822	0.2148	0.0981	0.9774	0.3293	Not-Rejected
Securitas	0.1956	0.2250	0.0729	0.9844	0.5501	Not-Rejected
Skanska	0.1396	0.1911	0.0491	0.9875	0.7966	Not-Rejected
SSAB	0.0300	0.2034	0.1090	0.9738	0.1914	Not-Rejected
SKF	-0.0610	0.4415	0.1920	0.9713	0.1488	Not-Rejected
Svenska Handelsbanken	0.1531	0.1878	0.0738	0.9643	0.0766	Not-Rejected
Swedbank	0.1863	0.1959	0.1137	0.9614	0.0535	Not-Rejected
Swedish Match	0.0874	0.2061	-0.0570	0.9871	0.7799	Not-Rejected
Tele2	-0.0882	0.2228	-0.1562	0.9216	0.0015	Rejected
TeliaSonera	-0.0352	0.1480	0.0344	0.9865	0.7452	Not-Rejected
Volvo Group	0.0651	0.2499	0.0536	0.9846	0.6509	Not-Rejected
Mean	0.0938	0.2345	0.0291			
Standard Deviation	0.0893	0.0694	0.1185			

Table 2 provides annualized summary statistics for 30 large-capitalization Swedish stock return time series, from a sample of monthly observations observed between October 2011 and November 2016, as well as the Wilk-Shapiro test statistic, each corresponding p-value, and the final decision on the null hypothesis for normality at the 5% level. The hypothesis of normality is rejected for level of p-value < 0.05

As it can be seen from the table, individual stocks exhibit a less moderate behaviour compared to country indexes, returns are slightly higher but with a larger standard deviation. Also, there is a much larger variability in skewness between the assets and normality is rejected for 9 stocks out of 30 at the 5% level. Out of these nine stocks, some of them exhibit positive skewness and some are exhibiting negative skewness. This is close to ideal for our analysis since the investors will have the opportunity to choose between stocks exhibiting significant different levels of skewness.

Please note, however, that stock prices of the current sample under analysis were observed in a period of bull markets across the developed world. As this might, ironically, skew our analysis, we have decided to enlarge the sample to include a broader period that would include stock prices behaviours during the 2007-08 financial crisis and report results for both. Unfortunately, Nokia hasn’t been traded on OMX for the full sample period and is therefore excluded without a replacement. We don’t see that this would have any significant impact on results since Nokia has the highest variance, if anything it would be bad for MVS-investors since Nokia also had the highest positive skewness.

5.1.2.3 OMX30 2003-2016

Table 3 – Summary statistics and normality testing for OMX 2003-16 monthly returns

OMX30	Expected Return	Standard Deviation	Skewness	W-test	p-value	H_0
ABB	0.1527	0.2414	-0.1195	0.9746	0.0069	Rejected
Alfa Laval	0.1578	0.2606	-0.1185	0.9883	0.1871	Not-Rejected
Assa Abloy	0.1697	0.2340	0.0532	0.9906	0.3286	Not-Rejected
AstraZeneca	0.0420	0.1969	0.0904	0.9860	0.1052	Not-Rejected
Atlas Copco (class A)	0.1835	0.2530	-0.1122	0.9779	0.0147	Rejected
Atlas Copco (class B)	0.1817	0.2571	-0.1188	0.9800	0.0242	Rejected
Electrolux	0.1384	0.3107	0.1802	0.9633	0.0007	Rejected
Ericsson	0.0147	0.3036	-0.0768	0.9622	0.0005	Rejected
Investor	0.1348	0.1993	-0.0948	0.9881	0.1808	Not-Rejected
Hennes & Mauritz	0.1055	0.1975	0.0712	0.9921	0.5493	Not-Rejected
Getinge	0.0901	0.2642	-0.0810	0.9815	0.0350	Rejected
Boliden	0.2734	0.4831	0.3367	0.9063	0.0000	Rejected
Kinnevik	0.1436	0.3049	-0.0897	0.9920	0.4536	Not-Rejected
Lundin Petroleum	0.2424	0.4068	0.2060	0.9676	0.0016	Rejected
Modern Times Group	0.0993	0.3549	0.1578	0.9244	0.0000	Rejected
Nordea	0.1021	0.2610	0.1030	0.9387	0.0000	Rejected
Sandvik	0.1043	0.2913	-0.1283	0.9825	0.0449	Rejected
SCA	0.0973	0.2214	0.0542	0.9822	0.0415	Rejected
SEB	0.0965	0.3052	-0.1407	0.8910	0.0000	Rejected
Securitas	0.0697	0.2424	-0.0785	0.9665	0.0013	Rejected
Skanska	0.1292	0.2619	0.1549	0.9385	0.0000	Rejected
SKF	0.1063	0.2640	0.1226	0.9799	0.0234	Rejected
SSAB	0.0667	0.3944	-0.1767	0.9836	0.0579	Not-Rejected
Svenska Handelsbanken	0.0968	0.2081	0.0523	0.9798	0.0231	Rejected
Swedbank	0.1091	0.3385	0.2360	0.8379	0.0000	Rejected
Swedish Match	0.1372	0.1733	-0.0824	0.9810	0.0308	Rejected
Tele2	-0.0037	0.2611	-0.1253	0.9777	0.0140	Rejected
TeliaSonera	0.0229	0.1981	-0.0567	0.9749	0.0075	Rejected
Volvo Group	0.1188	0.3213	-0.1388	0.9823	0.0423	Rejected
Mean	0.1166	0.2762	0.0028			
Standard Deviation	0.0598	0.0692	0.1369			

Table 3 provides annualized summary statistics for 29 large-capitalization Swedish stock return time series, from a sample of monthly observations observed between November 2003 and November 2016, as well as the Wilk-Shapiro test statistic, each corresponding p-value, and the final decision on the null hypothesis for normality at the 5% level. The hypothesis of normality is rejected for level of p-value < 0.05

In this table we have a longer time series of returns than in the previous one. As it can be seen from the table, returns and standard deviation are higher than in the shorter time period, whereas skewness seems quite the same. Further, normality is rejected for 22 stocks out of 29 at the 5% level, even though W-statistics are similar. This is mainly because we have a larger data sample which makes the observations of non-normality more certain. Just as the shorter sample, we observe both positive and negative skewness that will certainly have an impact on investors choices. However, the higher variance across assets within the sample might be large enough to offset the strongest preferences for skewness. Thus, we might see smaller differences between MV and MVS investors.

5.1.2.4 DAX30 2011-2016

Table 4 – Summary statistics and normality testing for DAX 2011-16 monthly returns

DAX30	Expected Return	Standard Deviation	Skewness	W-test	p-value	H_0
Adidas	0.2439	0.2395	-0.1158	0.9810	0.4074	Not-Rejected
Allianz	0.1458	0.2044	-0.0905	0.9804	0.4558	Not-Rejected
BASF	0.1035	0.2147	-0.0683	0.9838	0.6206	Not-Rejected
Bayer	0.1468	0.2051	-0.1052	0.9739	0.2354	Not-Rejected
Beiersdorf	0.1433	0.1679	0.0775	0.9835	0.6029	Not-Rejected
BMW	0.1129	0.2932	0.1276	0.9769	0.2737	Not-Rejected
Commerzbank	-0.0831	0.4104	0.2554	0.9241	0.0020	Rejected
Daimler	0.1761	0.2949	0.1473	0.9829	0.4892	Not-Rejected
Deutsche Bank	-0.0967	0.3443	-0.1373	0.9833	0.5945	Not-Rejected
Deutsche Börse	0.1020	0.1984	-0.0781	0.9611	0.0547	Not-Rejected
Deutsche Post	0.2064	0.1861	-0.1006	0.9760	0.2505	Not-Rejected
Deutsche Telekom	0.1057	0.1911	0.0921	0.9791	0.4031	Not-Rejected
E.ON	-0.1592	0.2951	-0.1354	0.9347	0.0048	Rejected
Fresenius	0.0913	0.3530	-0.3970	0.5194	0.0000	Rejected
Fresenius Medical Care	0.0894	0.1591	0.0795	0.9374	0.0061	Rejected
Heidelberg Cement	0.2139	0.1834	0.0283	0.9886	0.8561	Not-Rejected
Henkel	0.2163	0.1859	-0.0417	0.9733	0.1903	Not-Rejected
K+S	-0.0915	0.3583	-0.1605	0.9602	0.0504	Not-Rejected
Linde	0.0748	0.1947	-0.0727	0.9375	0.0061	Rejected
Deutsche Lufthansa	0.0866	0.3113	0.0687	0.9894	0.8872	Not-Rejected
MAN	0.0901	0.1699	0.0587	0.8168	0.0000	Rejected
Merck	0.1137	0.3245	-0.2973	0.7736	0.0000	Rejected
Metro	-0.0172	0.2904	-0.0922	0.9872	0.6987	Not-Rejected
Munich Re	0.1458	0.1810	-0.1000	0.9695	0.1451	Not-Rejected
RWE	-0.0879	0.3579	0.0482	0.9767	0.2687	Not-Rejected
Salzgitter	0.0061	0.3390	0.0857	0.9758	0.2864	Not-Rejected
SAP	0.1421	0.2119	0.1445	0.9295	0.0031	Rejected
Siemens	0.0821	0.1845	0.0806	0.9839	0.6244	Not-Rejected
ThyssenKrupp	0.0782	0.3398	-0.1495	0.9860	0.6357	Not-Rejected
Volkswagen Group	0.0931	0.3204	-0.2049	0.9316	0.0037	Rejected
Mean	0.0825	0.2570	-0.0351			
Standard Deviation	0.1002	0.0740	0.1397			

Table 4 provides annualized summary statistics for 30 large-capitalization German stock return time series, from a sample of monthly observations observed between October 2011 and November 2016, as well as the Wilk-Shapiro test statistic, each corresponding p-value, and the final decision on the null hypothesis for normality at the 5% level. The hypothesis of normality is rejected for level of p-value < 0.05

This data sample, even though another country, is very similar to the Swedish data sample from the same period of time. It is perhaps worth noticing that expected returns are slightly lower and standard deviations slightly higher. Also, skewness is less prominent but with similar variability between assets as the OMX30 from 2011-2016. Also here there are nine stocks that are significantly different from being normally distributed in their returns and they as well exhibits either positive or negative skewness. Thus, we can expect to see similar results as with the Swedish 2011-2016 sample, but maybe less differences between MV and MVS investors as variance is higher and lowering it through diversification might be more important than seeking a positively skewed portfolio.

5.1.3 Risk-free rate

An appropriate riskless asset for each sets of assets under analysis is introduced. For the sake of the analysis, the risk-free rate will be considered in its pure theoretical meaning: a guaranteed rate of return with zero variance, zero covariance with any of the risky assets, as well as zero skewness and zero coskewness with any other risky asset. Again, while this might be a bit of a stretch from a real-world standpoint, we felt that introducing a riskless asset in such form could be an optimal way to eliminate unnecessary complexity in our inputs derivation and subsequent analysis of results.

For our MSCI dataset, we will use the 2015 risk-free rate provided for by professor Kenneth French's online database, at 0.02%. We felt that this might be an adequate proxy for a generic riskless asset to the dataset under consideration.

For our Swedish stocks databases, the risk-free rate will be set at 0.2394%, the available interest rate for the 10-years Swedish government bonds, as of November 2016, while for our German stocks database, the risk-free rate will be set at 0.27%, the available interest rate for 10-years German government bonds as of November 2016. We felt that such rates might be an adequate proxy for the risk-free rate available to a respectively a Swedish or German investor, as they would reflect internal market condition and present no exchange rate risk. Again, interest rates per se are not of importance for our analysis, it is rather a tool to introduce the existence of a risk-free asset, as this means that the investors can decrease risk by investing in the risk-free instead of investing in just a risky portfolio, that is either mean-variance or mean-variance-skewness efficient.

For completeness, we will also run portfolio optimizations for all data samples without including a risk-free asset. However, because only the most risk-averse investors sometimes made use of the risk-free, the differences were small between the cases and most of these tables are put in the appendix instead.

5.1.4 General expectations

Overall, we expect larger differences between MV and MVS-investors when dealing with individual stocks rather than indexes. Indexes are diversified by definition, while the point of having preference for skewness is to hold a portfolio that is more concentrated of positive skewness assets, hence dealing with diversified assets rather than stocks might limit this possibility. However, diversification is also important to an MVS-investor which complicates the investment problem. There will be more clarity around this after the next section where we analyse how the investors aversions and preferences changes with the parameters in the utility function.

5.2 Parameter selection

The last thing left to be done before proceeding with the portfolio optimizations is to select the different set of parameters α, β and γ to define the utility function that we will try to maximize. The combinations between them will shape our utility function to fit comparable different levels and kinds of risk aversion.

To derive discrete sets of values for α, β and γ, we start looking at the values of $A(W)$, $R(W)$, defining the level of risk aversion, and their respective derivatives dA/dW and dR/dW that define the type of risk aversion, assuming $W = 1$. We start by creating a set of discrete values for $A(W)$ and $R(W)$, and we continue by choosing the signs of the derivatives to define different kinds of ARA and RRA. We then work backwards to find a combination of parameters α, β and γ that fits our requirements.

The set of feasible combinations can be endless. We chose to provide a handful of combinations that will provide us with a variety of different investors.

As we compute $A(W)$ and $R(W)$ at initial wealth level $W_0 = 1$, the two measures of risk aversion will have equal value, i.e. $A(W_0) = R(W_0)$. As this is the case, from here onwards the notation RA will be used to describe the numerical value of both.

We consider the values for RA in the interval $4 \leq RA \leq 16$, 16 being the highest level of risk aversion under consideration and 4 the lowest. We do not consider cases where $RA < 4$ as we have noticed that investors might allocate their whole wealth to the asset with the highest expected return, since they are not allowed to short-sell. This makes the choice unrealistic, and useless to our purposes, since both MVS and MV-investors would have the same allocation. On the other hand, investors with $RA > 16$ would increasingly display wildly negative levels of certainty equivalent for some samples in which there is no risk-free asset, this meaning that they would choose a certain loss over holding a portfolio of risky assets, as their aversion to risk is too high. We deemed levels of $RA > 16$ potentially unrealistic, and therefore we excluded them from the analysis. The tables below summarize the sets of parameters we choose to analyse for the three discrete values of RA of 4, 8 and 16.

Table 5 – Selection of parameters

Selection of parameters for RA=4										
Parameters			**Arrow-Pratt measures**			**Risk Aversion Classification**		**Kane Measures**		
α	**β**	**γ**	**RA**	**dA/dW**	**dR/dW**	**ARA**	**RRA**	**S(W)**	**K(W)**	**dS/dW**
0	4	1	4	0	4	CARA	IRRA	16	4.00	32
1	3	1	4	-1	3	DARA	IRRA	17	4.25	24
3	1	1	4	-3	1	DARA	IRRA	19	4.75	8
2	4	0.5	4	-3	1	DARA	IRRA	19	4.75	8.5
4	0	1	4	-4	0	DARA	CRRA	20	5.00	0
3	-1	-1	4	-5	-1	DARA	DRRA	21	5.25	-10
0	-4	-1	4	-8	-4	DARA	DRRA	24	6.00	-40
2	-1	-2	4	-8	-4	DARA	DRRA	24	6.00	-44
0	-2	-2	4	-12	-8	DARA	DRRA	28	7.00	-88
0	-1	-4	4	-20	-16	DARA	DRRA	36	9.00	-208

Table 5 summarizes the set of parameters we chose to analyse for RA=4, exhibiting corresponding derivatives, risk aversion classifications and Kane measures of skewness preference

Table 6 – Selection of parameters

Selection of parameters for RA=8										
Parameters			**Arrow-Pratt measures**			**Risk Aversion Classification**		**Kane Measures**		
α	**β**	**γ**	**RA**	**dA/dW**	**dR/dW**	**ARA**	**RRA**	**S(W)**	**K(W)**	**dS/dW**
0	8	1	8	0	8	CARA	IRRA	64	8.00	128
2	6	1	8	-2	6	DARA	IRRA	66	8.25	96
6	2	1	8	-6	2	DARA	IRRA	70	8.75	32
4	8	0.5	8	-6	2	DARA	IRRA	70	8.75	33
8	0	1	8	-8	0	DARA	CRRA	72	9.00	0
6	-2	-1	8	-10	-2	DARA	DRRA	74	9.25	-36
0	-8	-1	8	-16	-8	DARA	DRRA	80	10.00	-144
4	-2	-2	8	-16	-8	DARA	DRRA	80	10.00	-152
0	-4	-2	8	-24	-16	DARA	DRRA	88	11.00	-304
0	-2	-4	8	-40	-32	DARA	DRRA	104	13.00	-672

Table 6 summarizes the set of parameters we chose to analyse for RA=8, exhibiting corresponding derivatives, risk aversion classifications and Kane measures of skewness preference

Table 6 – Selection of parameters

Selection of parameters for RA=16										
Parameters			**Arrow-Pratt measures**			**Risk Aversion Classification**		**Kane Measures**		
α	**β**	**γ**	**RA**	**dA/dW**	**dR/dW**	**ARA**	**RRA**	**S(W)**	**K(W)**	**dS/dW**
0	16	1	16	0	16	CARA	IRRA	256	16.00	512
4	12	1	16	-4	12	DARA	IRRA	260	16.25	384
12	4	1	16	-12	4	DARA	IRRA	268	16.75	128
8	16	0.5	16	-12	4	DARA	IRRA	268	16.75	130
16	0	1	16	-16	0	DARA	CRRA	272	17.00	0
12	-4	-1	16	-20	-4	DARA	DRRA	276	17.25	-136
0	-16	-1	16	-32	-16	DARA	DRRA	288	18.00	-544
8	-4	-2	16	-32	-16	DARA	DRRA	288	18.00	-560
0	-8	-2	16	-48	-32	DARA	DRRA	304	19.00	-1120
0	-4	-4	16	-80	-64	DARA	DRRA	336	21.00	-2368

Table 7 summarizes the set of parameters we chose to analyse for RA=16, exhibiting corresponding derivatives, risk aversion classifications and Kane measures of skewness preference

It is possible to see that two patterns already emerge. First, we can see that the higher the level of risk aversion RA, the higher the values of relative skewness preference S, and skewness ratio K. Second the more negative dR/dW , the higher the values of S and K. Looking at $dR/dW = -16$ (a DRRA investor) for the different levels of RA we see that S increases by a factor of about 3 while looking at dR/dW = -16, -32 and -64 for the highest RA S increases only by some percent. Thus, more negative values in dR/dW (DRRA) seems to impact changes in S and K to a lesser extent than increases in the overall RA level. This means that investors with DRRA will have higher skewness preference than investors with CRRA or IRRA, everything else being equal, but a CARA investor who is more risk averse will probably have higher skewness preference and skewness ratio. However, there is an exception in our selection and that is that skewness ratio K decreases when moving from DRRA $RA = 4$ investor to a IRRA $RA = 8$ investor, but skewness preference increases.

In general, this is a predictor of our empirical results. As values of relative skewness preference and skewness ratio increase with the overall level of risk aversion, we expect a divergence of expected utility between the mean-variance and mean-variance-skewness optimized portfolios. Likewise, we expect such divergence to increase inversely with the value of dR/dW. It is unclear what happens when skewness preference and ratio moves in different directions.

5.3 Portfolio optimization

As now we have the full set of inputs necessary to our numerical analysis, we can finally optimize our MV and MVS portfolios that would solve the investor's allocation problem.

To find the optimal sets of weights for each optimization we use a numerical algorithm from a standard optimization toolbox. While a global optimum exists by definition, Athayde and Flores (2004) mention that in the MVS case, several local optimums might exist. When programming the optimization algorithm, one needs to make sure to search for a global optimum without settling for a local one. As such families of algorithms make use of the gradient of the function under optimization, the solution can be found considerably quicker if one is able to provide its analytical gradient.

To find the mean-variance optimal portfolio, we program the algorithm to find the set of weights that maximizes the Taylor series expansion of the expected utility, truncated after $m = 2$:

$$E[U(w)] \approx U(\bar{w}) - \bar{w}^{-\alpha-1}e^{-\beta\bar{w}^{\gamma}}(\alpha + \beta\gamma\bar{w}^{\gamma})\frac{\sigma_p^2}{2}$$

Also, to find the mean-variance-skewness optimal portfolio, we program the algorithm to find the set of weights that maximizes the Taylor series approximation truncated at $m = 3$:

$$\begin{aligned} E[U(w)] &\approx U(\bar{w}) \\ &\quad - \bar{w}^{-\alpha-1}e^{-\beta\bar{w}^{\gamma}}\Bigg((\alpha + \beta\gamma\bar{w}^{\gamma})\frac{\sigma_p^2}{2} \\ &\quad - \bar{w}^{-1}\big((\alpha + \beta\gamma\bar{w}^{\gamma})^2 + \alpha + \beta\gamma\bar{w}^{\gamma}(1-\gamma)\big)\frac{s_p^3}{6}\Bigg) \end{aligned}$$

We will then maximize both functions for each set of parameters, for each of the four datasets under analysis while constraining the weight to be greater or equal than zero, as short selling is not allowed. We will repeat the optimizations both with and without considering a riskless asset. Hence, for each dataset, we will then have one MV optimized portfolio and one MVS optimized portfolio for each set of parameter, from which we can extract the portfolio's expected return, standard deviation of return, skewness of return, Sharpe ratio and the investor's certainty equivalent.

5.4 Display and analysis of selected results

We will now display the results of the optimizations for each of the four datasets under analysis. As we realized that most results do not differ among cases in which the riskless asset is present or not, we have decided to focus our written analysis on the optimizations where a riskless asset is considered, as that would also allow us to observe each portfolio's Sharpe ratio. We will however comment on both cases for the OMX 2003-2016 sample, as we discovered a considerable difference between the two. Result tables for the remaining three samples in which the riskless asset is not considered will be shown in the appendix. All expected returns, standard deviations, skewnesses, certainty equivalents and Sharpe ratios are expressed on an annual basis.

5.4.1 MSCI 2011-2016

Figure 1 – Optimal portfolios in the MVS space for CARA/IRRA investor with RA=4

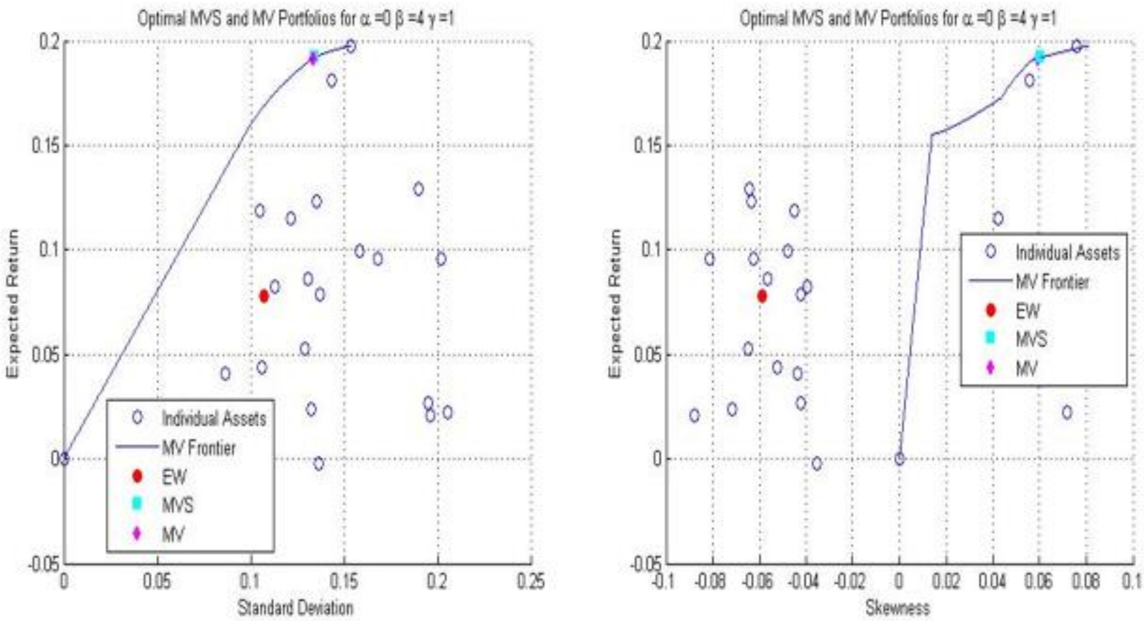

Figure 1 shows the results for a CARA/IRRA with RA=4 from the optimization of portfolio allocation using 20 MSCI indexes, from a sample of monthly observations between October 2011 and August 2016, plus a risk-free asset of 0.02% interest as the asset base. MV represents the mean-variance-optimized portfolio and MVS the mean-variance-skewness-optimized portfolio. Each graph in the figure is presenting the same 3D-graph from two different perspectives, an expected return-standard deviation perspective and an expected return-skewness perspective. For comparison there is also the equally-weighted (EW) portfolio, the individual assets and the mean-variance efficient frontier.

Figure 2 - Optimal portfolios in the MVS space for DARA/DRRA investor with RA=16

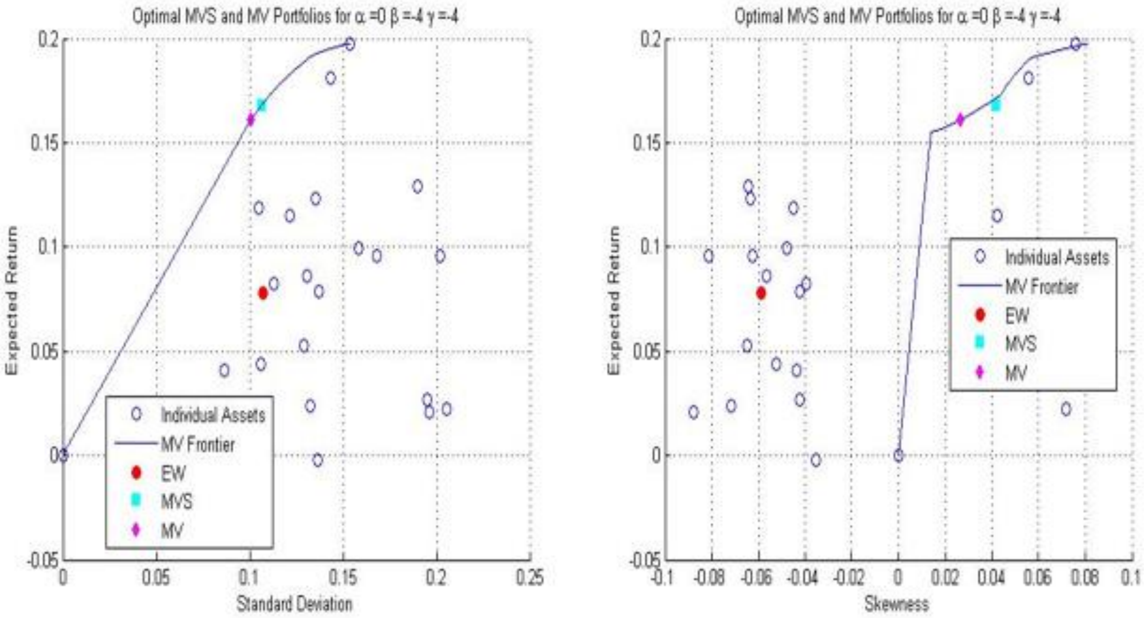

Figure 2 shows the results for a DARA/DRRA with RA=16 from the optimization of portfolio allocation using 20 MSCI indexes, from a sample of monthly observations between October 2011 and August 2016, plus a risk-free asset of 0.02% interest as the asset base. MV represents the mean-variance-optimized portfolio and MVS the mean-variance-skewness-optimized portfolio. Each graph in the figure is presenting the same 3D-graph from two different perspectives, an expected return-standard deviation perspective and an expected return-skewness perspective. For comparison there is also the equally-weighted (EW) portfolio, the individual assets and the mean-variance efficient frontier.

Table 8 summarizes the resulting different optimal portfolio moments for each investor whose preferences are reflected in their utility function, shaped by each different set of parameters, together with their corresponding certainty equivalents, the Euclidean distance between each portfolio, the Euclidean distance between each portfolio and the equally weighted portfolio EW, as well as the simple difference in certainty equivalent for each MVS and MV pair of portfolio for the same set of parameters. Figure 1 and Figure 2 illustrate the two extreme cases, the CARA/IRRA investor with RA=4 and the DARA/DRRA investor with RA=16. From each graph it is easy to see the efficient mean-variance frontier from two different angles, the classic view with the expected return on the Y-axis and the standard deviation on the X-axis, as well as an additional view that highlights the three-dimensionality of the space under consideration, showing the expected return on the Y-axis and skewness on the X-axis. Positions of the two different MV and MVS portfolios are shown for both cases and from both points of view, as well as the positions of the various assets under consideration and the position of the EW portfolio as a frame of reference.

The first pattern that we can see is that portfolio expected returns and standard deviations are increasing as we move from IRRA investors towards DRRA investors, for both

MV and MVS-investors. Naturally, those measures decreases when investors become more risk averse. Also, skewness increases for both MV and MVS-investors, but more for the MVS-investors. The reason for skewness to increase for MV-investors can be found in how the MV-efficient frontier happened to be for these assets, that is moving towards positive skewness when moving towards higher returns.

As we expected, we do not observe large differences, especially when looking at differences in certainty equivalents between MV and MVS investors. All investors under consideration seem to invest only in risky assets[2], as can be seen from the graphs that MVS and MV portfolios place towards the upper-right end of the mean-variance space. This is explained by the fact that only well diversified assets are available, which makes possible for even a risk averse individual to target a high return portfolio with low standard deviation. Further, the set of assets under analysis makes differences in skewness too small to exploit in any larger extent. It is worth noting, however, that the two assets with the highest return are also those assets with the most positive skewness which increases the observed differences.

As predicted from section 5.2, we observe a straightforward pattern of higher difference in CE with parameters that generate higher skewness preference and ratio[3], the highest difference being in the DARA/DRRA case, with RA=16, and the lowest being the CARA/IRRA at the other end of the spectrum, with RA=4. The only discrepancy is when skewness preference and ratio is moving in different directions when moving from an investor with DRRA and RA=4 to an investor with IRRA and RA=8.

Further, there are six pairs of investors who only have differences, within the pairs, in the derivative of skewness preference. For all of these pairs there is a small increase in the difference in CE for the investor with less negative or more positive derivative. The increase is true for both difference in percentage points as well as in percentages, but, as said, this difference is small and is nothing compared to the other differences.

When shifting focus to relative distances among portfolio weights as measured by the Euclidean norm, we observe that the pattern is very similar to that of the difference in CE.

[2] This makes Sharpe ratio and a case without a riskless asset redundant.

[3] See section 5.2 – Parameter selection

Table 8 – MSCI optimal portfolios with a riskless asset (MSCI 2011-16)

Arrow-Pratt Measures			Risk Type		Kane Measures			MVS-Optimized Portfolios					MV-Optimized Portfolios					Euclidean Distances			
RA	dA/dW	dR/dW	ARA	RRA	Skewness Preference	Skewness Ratio	dS/dW	Expected Return	Standard Deviation	Skewness	Certainty Equivalent	Sharpe Ratio	Expected Return	Standard Deviation	Skewness	Certainty Equivalent	Sharpe Ratio	MVS-MV	MVS-EW	MV-EW	Difference in CE
16	-80	-64	DARA	DRRA	336	21.00	-2368.0	0.1676	0.1065	0.0416	0.0903	1.5713	0.1610	0.1010	0.0266	0.0890	1.5924	0.1542	0.5369	0.4724	0.001348
16	-48	-32	DARA	DRRA	304	19.00	-1120.0	0.1667	0.1059	0.0413	0.0881	1.5729	0.1599	0.1002	0.0248	0.0870	1.5942	0.1681	0.5357	0.4683	0.001111
16	-32	-16	DARA	DRRA	288	18.00	-544.0	0.1663	0.1056	0.0412	0.0870	1.5735	0.1594	0.0998	0.0239	0.0860	1.5949	0.1749	0.5353	0.4665	0.000974
16	-32	-16	DARA	DRRA	288	18.00	-560.0	0.1663	0.1056	0.0412	0.0870	1.5735	0.1594	0.0998	0.0239	0.0860	1.5949	0.1750	0.5353	0.4665	0.000972
16	-20	-4	DARA	DRRA	276	17.25	-136.0	0.1659	0.1053	0.0409	0.0862	1.5743	0.1591	0.0996	0.0232	0.0853	1.5954	0.1780	0.5340	0.4652	0.000863
16	-16	0	DARA	CRRA	272	17.00	0.0	0.1656	0.1050	0.0403	0.0859	1.5756	0.1589	0.0995	0.0230	0.0850	1.5956	0.1726	0.5303	0.4648	0.000827
16	-12	4	DARA	IRRA	268	16.75	128.0	0.1652	0.1046	0.0398	0.0856	1.5769	0.1588	0.0994	0.0228	0.0848	1.5957	0.1674	0.5268	0.4644	0.000792
16	-12	4	DARA	IRRA	268	16.75	130.0	0.1652	0.1046	0.0398	0.0856	1.5769	0.1588	0.0994	0.0228	0.0848	1.5957	0.1675	0.5268	0.4644	0.000793
16	-4	12	DARA	IRRA	260	16.25	384.0	0.1645	0.1040	0.0387	0.0851	1.5793	0.1586	0.0992	0.0223	0.0843	1.5960	0.1577	0.5203	0.4637	0.000728
16	0	16	CARA	IRRA	256	16.00	512.0	0.1642	0.1037	0.0382	0.0848	1.5804	0.1584	0.0991	0.0221	0.0841	1.5962	0.1532	0.5173	0.4633	0.000698
8	-40	-32	DARA	DRRA	104	13.00	-672.0	0.1850	0.1246	0.0528	0.1307	1.4837	0.1817	0.1204	0.0494	0.1305	1.5067	0.0681	0.6294	0.6008	0.000170
8	-24	-16	DARA	DRRA	88	11.00	-304.0	0.1830	0.1220	0.0510	0.1288	1.4977	0.1805	0.1190	0.0485	0.1287	1.5145	0.0519	0.6121	0.5920	0.000107
8	-16	-8	DARA	DRRA	80	10.00	-144.0	0.1821	0.1209	0.0502	0.1279	1.5040	0.1799	0.1183	0.0481	0.1278	1.5182	0.0450	0.6047	0.5880	0.000084
8	-16	-8	DARA	DRRA	80	10.00	-152.0	0.1820	0.1209	0.0502	0.1279	1.5040	0.1799	0.1183	0.0481	0.1278	1.5182	0.0449	0.6046	0.5879	0.000083
8	-10	-2	DARA	DRRA	74	9.25	-36.0	0.1814	0.1201	0.0496	0.1272	1.5084	0.1794	0.1178	0.0478	0.1271	1.5209	0.0402	0.5996	0.5851	0.000069
8	-8	0	DARA	CRRA	72	9.00	0.0	0.1812	0.1199	0.0495	0.1269	1.5098	0.1793	0.1177	0.0477	0.1269	1.5217	0.0387	0.5980	0.5841	0.000064
8	-6	2	DARA	IRRA	70	8.75	32.0	0.1810	0.1196	0.0493	0.1267	1.5112	0.1791	0.1175	0.0476	0.1266	1.5226	0.0372	0.5964	0.5832	0.000060
8	-6	2	DARA	IRRA	70	8.75	33.0	0.1810	0.1196	0.0493	0.1267	1.5111	0.1791	0.1175	0.0476	0.1266	1.5226	0.0372	0.5964	0.5832	0.000060
8	-2	6	DARA	IRRA	66	8.25	96.0	0.1805	0.1191	0.0489	0.1263	1.5138	0.1789	0.1172	0.0474	0.1262	1.5243	0.0344	0.5933	0.5814	0.000052
8	0	8	CARA	IRRA	64	8.00	128.0	0.1803	0.1189	0.0488	0.1260	1.5151	0.1787	0.1171	0.0473	0.1260	1.5251	0.0330	0.5918	0.5805	0.000049
4	-20	-16	DARA	DRRA	36	9.00	-208.0	0.1927	0.1358	0.0628	0.1601	1.4175	0.1919	0.1341	0.0604	0.1600	1.4297	0.0641	0.7251	0.7037	0.000067
4	-12	-8	DARA	DRRA	28	7.00	-88.0	0.1923	0.1350	0.0618	0.1588	1.4231	0.1918	0.1338	0.0600	0.1587	1.4315	0.0477	0.7153	0.7005	0.000039
4	-8	-4	DARA	DRRA	24	6.00	-40.0	0.1922	0.1347	0.0613	0.1581	1.4255	0.1917	0.1337	0.0598	0.1581	1.4324	0.0402	0.7109	0.6990	0.000029
4	-8	-4	DARA	DRRA	24	6.00	-44.0	0.1922	0.1347	0.0613	0.1581	1.4256	0.1917	0.1337	0.0598	0.1581	1.4324	0.0400	0.7109	0.6989	0.000029
4	-5	-1	DARA	DRRA	21	5.25	-10.0	0.1921	0.1344	0.0609	0.1576	1.4273	0.1917	0.1336	0.0597	0.1576	1.4330	0.0347	0.7079	0.6979	0.000022
4	-4	0	DARA	CRRA	20	5.00	0.0	0.1921	0.1344	0.0608	0.1574	1.4278	0.1917	0.1336	0.0596	0.1574	1.4332	0.0330	0.7070	0.6975	0.000020
4	-3	1	DARA	IRRA	19	4.75	8.0	0.1920	0.1343	0.0607	0.1573	1.4283	0.1917	0.1336	0.0596	0.1572	1.4334	0.0312	0.7060	0.6971	0.000018
4	-3	1	DARA	IRRA	19	4.75	8.5	0.1920	0.1343	0.0607	0.1573	1.4283	0.1917	0.1336	0.0596	0.1572	1.4334	0.0312	0.7060	0.6971	0.000018
4	-1	3	DARA	IRRA	17	4.25	24.0	0.1919	0.1341	0.0605	0.1569	1.4294	0.1916	0.1335	0.0595	0.1569	1.4338	0.0278	0.7042	0.6965	0.000015
4	0	4	CARA	IRRA	16	4.00	32.0	0.1919	0.1341	0.0604	0.1568	1.4299	0.1916	0.1335	0.0594	0.1567	1.4340	0.0261	0.7033	0.6961	0.000013

Table 8 provides the result from the optimization of portfolio allocation using 20 MSCI indexes, from a sample of monthly observations between October 2011 and August 2016, plus a risk-free asset of 0.02% interest as the asset base. Each row corresponds to one specific investor defined by its risk aversion (RA), derivative of absolute risk aversion (dA/dW), derivative of relative risk aversion (dR/dW), skewness preference, skewness ratio, and derivative of skewness preference (dS/dW). For simplicity the type of risk aversion is included, with ARA meaning absolute risk aversion and RRA meaning relative risk aversion. These can be either increasing, constant or decreasing. For each investor there are two resulting portfolios, one when the investor considers mean, variance and skewness, and one when the investor only consider mean and variance. The portfolio moments and certainty equivalent are annualized. Euclidean distances are computed on the difference in the weights in assets between the MVS-optimized portfolio, the MV-optimized portfolio and the equally weighted portfolio. Difference in certainty equivalent (CE) is calculated as the CE of the MVS-portfolio less the CE of the MV-portfolio for each investor.

5.4.2 OMX30 2011-2016

Figure 3 - Optimal portfolios in the MVS space for CARA/IRRA investor with RA=4

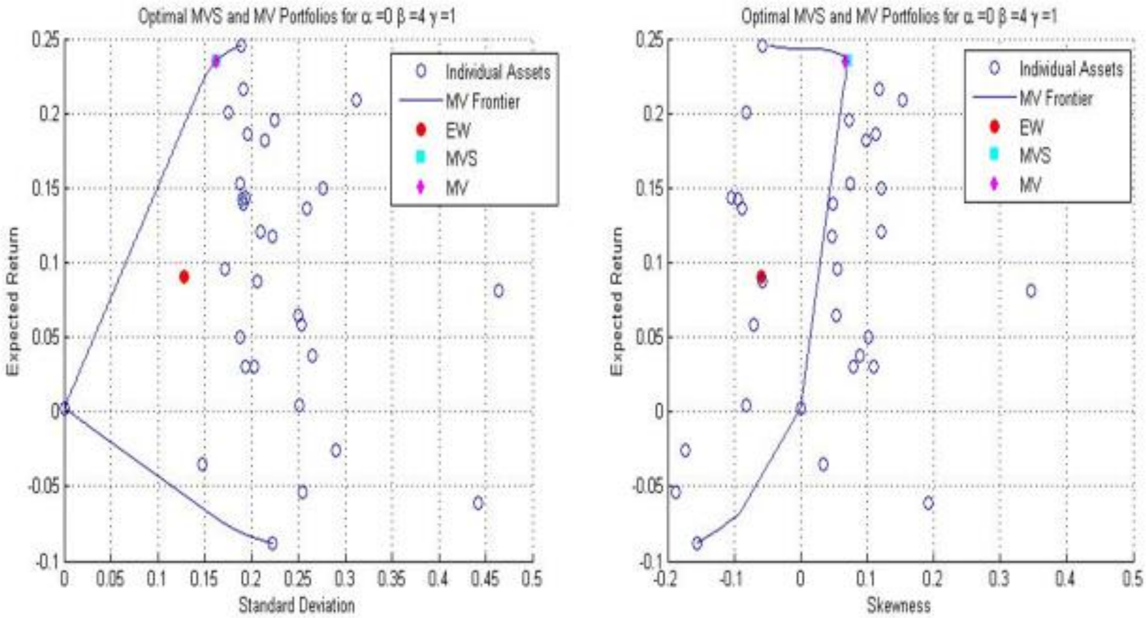

Figure 3 shows the results for a CARA/IRRA with RA=4 from the optimization of portfolio allocation using OMXS30 (large-capitalization Swedish stocks), from a sample of monthly observations between October 2011 and November 2016, plus a risk-free asset of 0.24% interest as the asset base. MV represents the mean-variance-optimized portfolio and MVS the mean-variance-skewness-optimized portfolio. Each graph in the figure is presenting the same 3D-graph from two different perspectives, an expected return-standard deviation perspective and an expected return-skewness perspective. For comparison there is also the equally-weighted (EW) portfolio, the individual assets and the mean-variance efficient frontier.

Figure 4 - Optimal portfolios in the MVS space for DARA/DRRA investor with RA=16

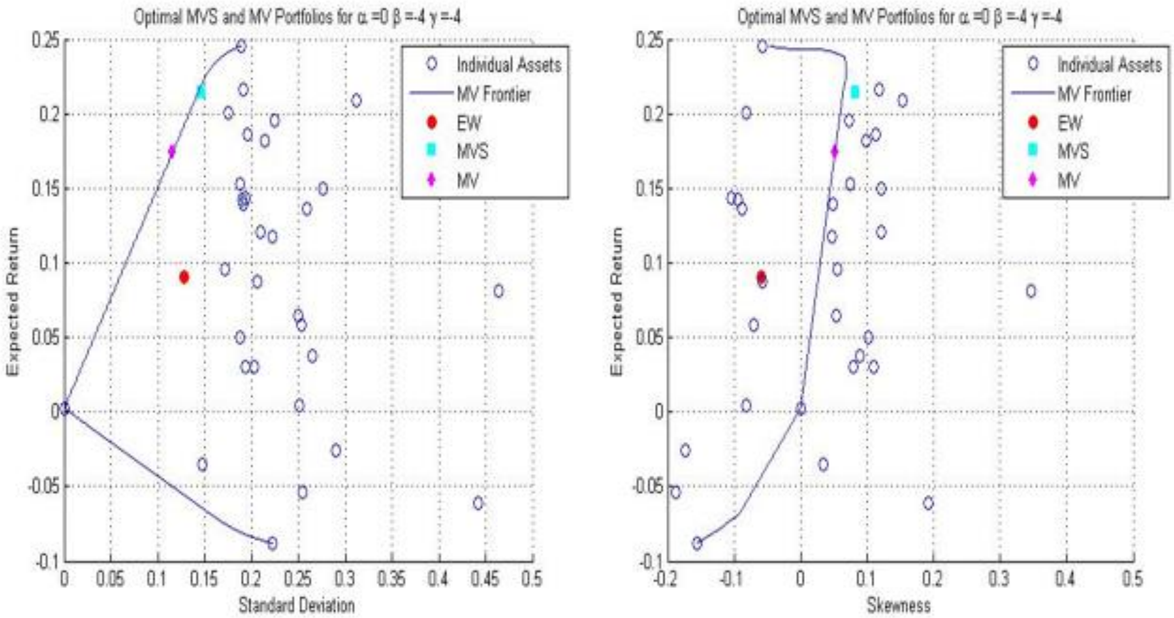

Figure 4 shows the results for a DARA/DRRA with RA=16 from the optimization of portfolio allocation using OMXS30 (large-capitalization Swedish stocks), from a sample of monthly observations between October 2011 and November 2016, plus a risk-free asset of 0.24% interest as the asset base. MV represents the mean-variance-optimized portfolio and MVS the mean-variance-skewness-optimized portfolio. Each graph in the figure is presenting the same 3D-graph from two different perspectives, an expected return-standard deviation perspective and an expected return-skewness perspective. For comparison there is also the equally-weighted (EW) portfolio, the individual assets and the mean-variance efficient frontier.

From Table 9, figure 3 and figure 4 we can see that the overall difference in certainty equivalents and distance among MV and MVS portfolios is much higher than in the previously analysed sample. As stocks are more spread out in terms of skewness, MVS investors with a strong preference for skewness will choose a very different portfolio compared to MV investors, and we can see a notable peak of difference in CE of 0.73% per year in the DARA/DRRA case with RA=16, graphically visualized in figure 4.

When comparing differences in CE and distance, we observe a similar pattern to that previously analysed MSCI results. Since the variance is higher than in the MSCI case, the most risk averse investors will have some weight in the riskless asset and we can therefore compare Sharpe ratios. It is thus possible to conclude that the MVS investors are clearly deviating from the theorem that says that efficient portfolios should have maximum Sharpe ratio and MVS-portfolios are thus suboptimal in a MV framework. The MVS-portfolios also breaks the two fund separation theorem, since different SR means a different tangency portfolio. It is important to note that the Sharpe ratio decreases when we have MV-investors with lower RA because they don't invest anything in the riskless asset, because of the no-short constraint.

Additionally, when looking at expected portfolio return and standard deviation we again observe that those measures increases when moving from IRRA to DRRA investors. However, skewness increases for MVS-investors but decreases for MV-investors when RA=4. This behaviour of MV-investors is once again due to the shape of the MV-efficient frontier, whereas it is not as asset dependent for MVS-investors.

Table 9 - Swedish large cap optimal portfolios with a riskless asset (OMX 2011-16)

Arrow-Pratt Measures			Risk Type		Kane Measures			MVS-Optimized Portfolios					MV-Optimized Portfolios					Euclidean Distances			
RA	**dA/dW**	**dR/dW**	**ARA**	**RRA**	**Skewness Preference**	**Skewness Ratio**	**dS/dW**	**Expected Return**	**Standard Deviation**	**Skewness**	**Certainty Equivalent**	**Sharpe Ratio**	**Expected Return**	**Standard Deviation**	**Skewness**	**Certainty Equivalent**	**Sharpe Ratio**	**MVS-MV**	**MVS-EW**	**MV-EW**	**Difference in CE**
16	-80	-64	DARA	DRRA	336	21.00	-2368.0	0.2142	0.1469	0.0831	0.0932	1.4423	0.1742	0.1160	0.0516	0.0859	1.4815	0.3242	0.5371	0.4294	0.007298
16	-48	-32	DARA	DRRA	304	19.00	-1120.0	0.2130	0.1455	0.0815	0.0878	1.4479	0.1657	0.1102	0.0491	0.0826	1.4815	0.3421	0.5239	0.4291	0.005202
16	-32	-16	DARA	DRRA	288	18.00	-544.0	0.2074	0.1411	0.0781	0.0851	1.4527	0.1619	0.1077	0.0479	0.0811	1.4815	0.3201	0.5023	0.4306	0.004005
16	-32	-16	DARA	DRRA	288	18.00	-560.0	0.2071	0.1409	0.0779	0.0850	1.4529	0.1619	0.1076	0.0479	0.0810	1.4815	0.3182	0.5013	0.4306	0.003987
16	-20	-4	DARA	DRRA	276	17.25	-136.0	0.1960	0.1327	0.0721	0.0833	1.4590	0.1592	0.1059	0.0471	0.0800	1.4815	0.2608	0.4717	0.4321	0.003295
16	-16	0	DARA	CRRA	272	17.00	0.0	0.1930	0.1305	0.0706	0.0827	1.4605	0.1584	0.1053	0.0469	0.0796	1.4815	0.2461	0.4651	0.4327	0.003102
16	-12	4	DARA	IRRA	268	16.75	128.0	0.1902	0.1285	0.0692	0.0822	1.4619	0.1576	0.1047	0.0466	0.0793	1.4815	0.2326	0.4596	0.4333	0.002920
16	-12	4	DARA	IRRA	268	16.75	130.0	0.1902	0.1285	0.0692	0.0822	1.4618	0.1576	0.1047	0.0466	0.0793	1.4815	0.2327	0.4596	0.4333	0.002921
16	-4	12	DARA	IRRA	260	16.25	384.0	0.1852	0.1248	0.0666	0.0812	1.4642	0.1559	0.1036	0.0461	0.0786	1.4815	0.2095	0.4513	0.4347	0.002600
16	0	16	CARA	IRRA	256	16.00	512.0	0.1829	0.1232	0.0655	0.0808	1.4652	0.1551	0.1031	0.0459	0.0783	1.4815	0.1996	0.4482	0.4354	0.002458
8	-40	-32	DARA	DRRA	104	13.00	-672.0	0.2262	0.1532	0.0774	0.1494	1.4613	0.2261	0.1521	0.0684	0.1484	1.4710	0.1248	0.5905	0.5881	0.000989
8	-24	-16	DARA	DRRA	88	11.00	-304.0	0.2256	0.1523	0.0761	0.1458	1.4657	0.2254	0.1514	0.0683	0.1450	1.4728	0.1037	0.5812	0.5786	0.000776
8	-16	-8	DARA	DRRA	80	10.00	-144.0	0.2253	0.1519	0.0754	0.1439	1.4675	0.2251	0.1511	0.0682	0.1432	1.4736	0.0939	0.5770	0.5741	0.000678
8	-16	-8	DARA	DRRA	80	10.00	-152.0	0.2253	0.1519	0.0754	0.1439	1.4676	0.2251	0.1511	0.0682	0.1432	1.4736	0.0937	0.5769	0.5740	0.000676
8	-10	-2	DARA	DRRA	74	9.25	-36.0	0.2250	0.1516	0.0749	0.1425	1.4688	0.2248	0.1509	0.0681	0.1419	1.4742	0.0869	0.5739	0.5707	0.000607
8	-8	0	DARA	CRRA	72	9.00	0.0	0.2250	0.1515	0.0748	0.1420	1.4692	0.2247	0.1508	0.0681	0.1414	1.4744	0.0846	0.5730	0.5696	0.000585
8	-6	2	DARA	IRRA	70	8.75	32.0	0.2249	0.1514	0.0746	0.1415	1.4696	0.2247	0.1507	0.0681	0.1410	1.4745	0.0824	0.5720	0.5686	0.000563
8	-6	2	DARA	IRRA	70	8.75	33.0	0.2249	0.1514	0.0746	0.1415	1.4696	0.2247	0.1507	0.0681	0.1410	1.4745	0.0824	0.5720	0.5686	0.000563
8	-2	6	DARA	IRRA	66	8.25	96.0	0.2248	0.1512	0.0743	0.1406	1.4704	0.2245	0.1506	0.0680	0.1400	1.4749	0.0780	0.5701	0.5664	0.000519
8	0	8	CARA	IRRA	64	8.00	128.0	0.2247	0.1511	0.0742	0.1401	1.4707	0.2244	0.1505	0.0680	0.1396	1.4750	0.0759	0.5691	0.5654	0.000498
4	-20	-16	DARA	DRRA	36	9.00	-208.0	0.2359	0.1651	0.0734	0.1888	1.4140	0.2368	0.1664	0.0664	0.1885	1.4093	0.0883	0.7242	0.7451	0.000265
4	-12	-8	DARA	DRRA	28	7.00	-88.0	0.2356	0.1646	0.0726	0.1865	1.4170	0.2360	0.1649	0.0668	0.1863	1.4167	0.0737	0.7203	0.7312	0.000169
4	-8	-4	DARA	DRRA	24	6.00	-40.0	0.2352	0.1639	0.0718	0.1853	1.4206	0.2355	0.1642	0.0669	0.1852	1.4201	0.0626	0.7149	0.7246	0.000126
4	-8	-4	DARA	DRRA	24	6.00	-44.0	0.2352	0.1639	0.0718	0.1853	1.4206	0.2355	0.1642	0.0669	0.1852	1.4202	0.0624	0.7148	0.7245	0.000125
4	-5	-1	DARA	DRRA	21	5.25	-10.0	0.2350	0.1634	0.0713	0.1845	1.4231	0.2352	0.1637	0.0670	0.1844	1.4226	0.0546	0.7111	0.7198	0.000098
4	-4	0	DARA	CRRA	20	5.00	0.0	0.2349	0.1633	0.0711	0.1842	1.4240	0.2351	0.1635	0.0671	0.1841	1.4234	0.0520	0.7098	0.7182	0.000089
4	-3	1	DARA	IRRA	19	4.75	8.0	0.2348	0.1631	0.0710	0.1839	1.4248	0.2350	0.1633	0.0671	0.1838	1.4242	0.0494	0.7086	0.7166	0.000081
4	-3	1	DARA	IRRA	19	4.75	8.5	0.2348	0.1631	0.0710	0.1839	1.4248	0.2350	0.1633	0.0671	0.1838	1.4242	0.0494	0.7086	0.7166	0.000081
4	-1	3	DARA	IRRA	17	4.25	24.0	0.2346	0.1628	0.0706	0.1833	1.4264	0.2348	0.1630	0.0672	0.1832	1.4257	0.0442	0.7062	0.7136	0.000066
4	0	4	CARA	IRRA	16	4.00	32.0	0.2345	0.1627	0.0705	0.1830	1.4271	0.2347	0.1629	0.0672	0.1829	1.4265	0.0417	0.7051	0.7121	0.000059

Table 9 provides the result from the optimization of portfolio allocation using OMX30 (large-capitalization Swedish stocks), from a sample of monthly observations between October 2011 and November 2016, plus a risk-free asset of 0.24% interest as the asset base. Each row corresponds to one specific investor defined by its risk aversion (RA), derivative of absolute risk aversion (dA/dW), derivative of relative risk aversion (dR/dW), skewness preference, skewness ratio, and derivative of skewness preference (dS/dW). For simplicity the type of risk aversion is included, with ARA meaning absolute risk aversion and RRA meaning relative risk aversion. These can be either increasing, constant or decreasing. For each investor there are two resulting portfolios, one when the investor considers mean, variance and skewness, and one when the investor only consider mean and variance. The portfolio moments and certainty equivalent are annualized. Euclidean distances are computed on the difference in the weights in assets between the MVS-optimized portfolio, the MV-optimized portfolio and the equally weighted portfolio. Difference in certainty equivalent (CE) is calculated as the CE of the MVS-portfolio less the CE of the MV-portfolio for each investor.

5.4.3 OMX30 2003-2016

5.4.3.1 With risk-free

Figure 5 - Optimal portfolios in the MVS space for CARA/IRRA investor with RA=4

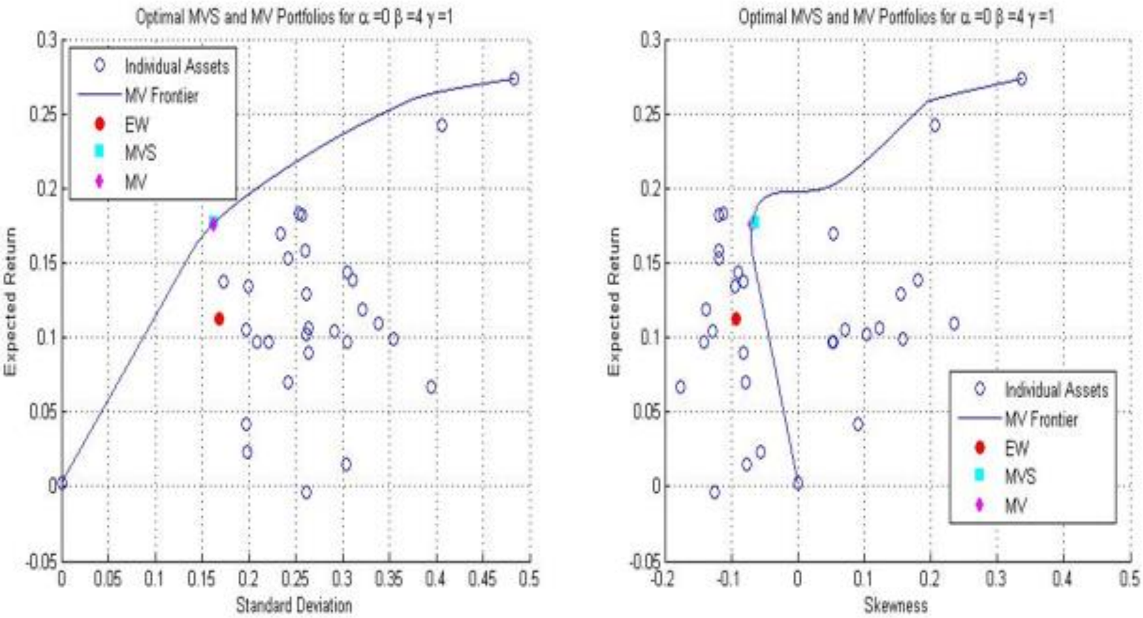

Figure 5 shows the results for a CARA/IRRA with RA=4 from the optimization of portfolio allocation using OMXS30 (large-capitalization Swedish stocks) excluding Nokia, from a sample of monthly observations between October 2003 and November 2016, plus a risk-free asset of 0.24% interest as the asset base. MV represents the mean-variance-optimized portfolio and MVS the mean-variance-skewness-optimized portfolio. Each graph in the figure is presenting the same 3D-graph from two different perspectives, an expected return-standard deviation perspective and an expected return-skewness perspective. For comparison there is also the equally-weighted (EW) portfolio, the individual assets and the mean-variance efficient frontier.

Figure 6 - Optimal portfolios in the MVS space for DARA/DRRA investor with RA=16

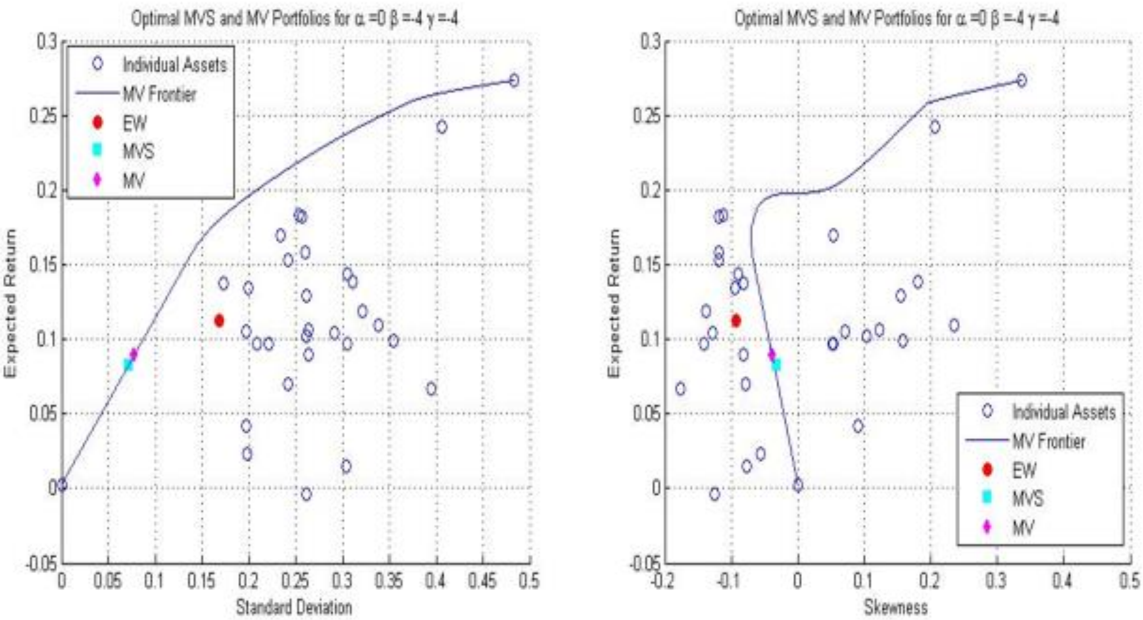

Figure 6 shows the results for a DARA/DRRA with RA=16 from the optimization of portfolio allocation using OMXS30 (large-capitalization Swedish stocks) excluding Nokia, from a sample of monthly observations between October 2003 and November 2016, plus a risk-free asset of 0.24% interest as the asset base. MV represents the mean-variance-optimized portfolio and MVS the mean-variance-skewness-optimized portfolio. Each graph in the figure is presenting the same 3D-graph from two different perspectives, an expected return-standard deviation perspective and an expected return-skewness perspective. For comparison there is also the equally-weighted (EW) portfolio, the individual assets and the mean-variance efficient frontier.

To add some robustness to the bull market case of OMX30 we also looked at a longer sample period to include stock returns behaviours during the financial crisis. This sample exhibits higher average returns and standard deviation compared to the previous shorter sample, whereas skewness levels look similar. After the optimizations however, expected portfolio returns turn out to be generally lower and the portfolios with the highest standard deviation are in the same ballpark as those that emerged using inputs from the 2011-2016 sample. This means that investors have needed to trade-off a potential higher expected return to contrast a more prominent standard deviation through diversification. The additional need for diversification has also lead to more negative overall portfolio skewness across our results and a smaller distance between MV and MVS portfolios according to Euclidean norm.

Because the expected return is low compared to standard deviation the investors put quite some weight in the risk-free asset which makes the differences between MV and MVS-investors even smaller, in line with the comments by Jondeau and Rockinger (2006). The differences are further diminished because the MV-frontier is moving towards positive skewness when decreasing variance, which means that two of the preferences are fulfilled simultaneously. Moreover, since MVS-investors are not moving further away from the MV efficient frontier, there is not a viable opportunity to gain portfolio positive skewness by holding a less-diversified portfolio.

Further, when comparing differences in CE and distances the pattern of differences across the spectrum of risk aversion is very similar to the what we observed using the previous OMX30 sample. There are however some exceptions. When we move from a DRRA investor with RA=8 to an IRRA investor with RA=16 both the differences in certain equivalent and portfolios distances are dropping, even though both skewness preferences and skewness ratios are increasing. This is potentially explained by investors' tendency to allocate more wealth in the risk-free asset, which decreases overall differences between MV and MVS. This is somewhat supported because the drop is less pronounced and less persistent if we look at percentage differences in CE. This is because CE decreases a lot for both MV and MVS-investors when moving from RA=8 to RA=16 investors. The other alternatives are that this is asset base specific or that skewness preference and ratio is missing some information about MVS-investors.

Although overall differences have decreased, it is worth noticing that Sharpe ratios of the very risk averse MVS-investors are inferior to those of MV-investors meaning that those investors' optimal portfolio is suboptimal from a MV perspective, in line with the findings of Mitton and Vorkink (2007). Further, different MVS investors hold portfolios which are exhibiting different Sharpe ratios, in clear contrast with the prediction of two fund separation theorem.

Moreover, here we do have a case in the upper part of the table when skewness is getting more negative for MVS-investors when moving from IRRA-investors to DRRA-investors. This shows that the general MVS choice is strongly dependent on the underlying characteristics of specific asset available in the investment universe, whether it is worth moving towards positive skewness since the opportunities to increase expected return might be better. Although not positive, skewness in MVS optimal portfolios is still less negative than in the MV optimized portfolios.

Table 10 - Swedish large cap optimal portfolios with a riskless asset (OMX 2003-16)

Arrow-Pratt Measures			Risk Type		Kane Measures			MVS-Optimized Portfolios					MV-Optimized Portfolios					Euclidean Distances			
RA	dA/dW	dR/dW	ARA	RRA	Skewness Preference	Skewness Ratio	dS/dW	Expected Return	Standard Deviation	Skewness	Certainty Equivalent	Sharpe Ratio	Expected Return	Standard Deviation	Skewness	Certainty Equivalent	Sharpe Ratio	MVS-MV	MVS-EW	MV-EW	Difference in CE
16	-80	-64	DARA	DRRA	336	21.00	-2368.0	0.0828	0.0724	-0.0321	0.0417	1.1105	0.0895	0.0781	-0.0390	0.0409	1.1153	0.0617	0.5180	0.4867	0.000726
16	-48	-32	DARA	DRRA	304	19.00	-1120.0	0.0815	0.0712	-0.0320	0.0413	1.1114	0.0875	0.0763	-0.0381	0.0407	1.1153	0.0546	0.5229	0.4946	0.000573
16	-32	-16	DARA	DRRA	288	18.00	-544.0	0.0809	0.0706	-0.0319	0.0411	1.1118	0.0865	0.0754	-0.0376	0.0406	1.1153	0.0512	0.5253	0.4985	0.000506
16	-32	-16	DARA	DRRA	288	18.00	-560.0	0.0809	0.0706	-0.0319	0.0411	1.1118	0.0865	0.0754	-0.0376	0.0406	1.1153	0.0511	0.5253	0.4985	0.000506
16	-20	-4	DARA	DRRA	276	17.25	-136.0	0.0805	0.0702	-0.0319	0.0410	1.1121	0.0858	0.0748	-0.0373	0.0405	1.1153	0.0487	0.5271	0.5014	0.000460
16	-16	0	DARA	CRRA	272	17.00	0.0	0.0803	0.0701	-0.0319	0.0409	1.1122	0.0856	0.0746	-0.0372	0.0405	1.1153	0.0479	0.5277	0.5023	0.000446
16	-12	4	DARA	IRRA	268	16.75	128.0	0.0802	0.0699	-0.0318	0.0409	1.1123	0.0853	0.0744	-0.0371	0.0404	1.1153	0.0471	0.5282	0.5033	0.000432
16	-12	4	DARA	IRRA	268	16.75	130.0	0.0802	0.0699	-0.0318	0.0409	1.1123	0.0853	0.0744	-0.0371	0.0404	1.1153	0.0471	0.5282	0.5033	0.000432
16	-4	12	DARA	IRRA	260	16.25	384.0	0.0799	0.0697	-0.0318	0.0408	1.1124	0.0849	0.0740	-0.0369	0.0404	1.1153	0.0455	0.5294	0.5052	0.000404
16	0	16	CARA	IRRA	256	16.00	512.0	0.0798	0.0695	-0.0318	0.0407	1.1125	0.0847	0.0738	-0.0368	0.0403	1.1153	0.0447	0.5300	0.5061	0.000391
8	-40	-32	DARA	DRRA	104	13.00	-672.0	0.1597	0.1421	-0.0606	0.0826	1.1073	0.1593	0.1409	-0.0692	0.0818	1.1137	0.0757	0.4995	0.4814	0.000837
8	-24	-16	DARA	DRRA	88	11.00	-304.0	0.1584	0.1406	-0.0619	0.0811	1.1099	0.1585	0.1401	-0.0691	0.0805	1.1144	0.0639	0.4945	0.4809	0.000622
8	-16	-8	DARA	DRRA	80	10.00	-144.0	0.1577	0.1398	-0.0624	0.0804	1.1109	0.1581	0.1397	-0.0691	0.0799	1.1147	0.0582	0.4913	0.4807	0.000530
8	-16	-8	DARA	DRRA	80	10.00	-152.0	0.1577	0.1398	-0.0624	0.0804	1.1109	0.1581	0.1397	-0.0691	0.0799	1.1147	0.0581	0.4913	0.4807	0.000528
8	-10	-2	DARA	DRRA	74	9.25	-36.0	0.1572	0.1392	-0.0627	0.0799	1.1116	0.1578	0.1394	-0.0691	0.0794	1.1148	0.0541	0.4890	0.4805	0.000467
8	-8	0	DARA	CRRA	72	9.00	0.0	0.1570	0.1390	-0.0628	0.0797	1.1118	0.1577	0.1393	-0.0691	0.0793	1.1149	0.0528	0.4882	0.4805	0.000448
8	-6	2	DARA	IRRA	70	8.75	32.0	0.1568	0.1389	-0.0630	0.0796	1.1120	0.1576	0.1392	-0.0691	0.0791	1.1149	0.0515	0.4875	0.4805	0.000429
8	-6	2	DARA	IRRA	70	8.75	33.0	0.1568	0.1389	-0.0630	0.0796	1.1120	0.1576	0.1392	-0.0691	0.0791	1.1149	0.0515	0.4875	0.4805	0.000429
8	-2	6	DARA	IRRA	66	8.25	96.0	0.1565	0.1385	-0.0632	0.0792	1.1124	0.1574	0.1390	-0.0690	0.0788	1.1150	0.0490	0.4860	0.4804	0.000394
8	0	8	CARA	IRRA	64	8.00	128.0	0.1563	0.1383	-0.0633	0.0791	1.1125	0.1573	0.1390	-0.0690	0.0787	1.1151	0.0478	0.4853	0.4804	0.000377
4	-20	-16	DARA	DRRA	36	9.00	-208.0	0.1801	0.1694	-0.0590	0.1262	1.0492	0.1790	0.1671	-0.0676	0.1259	1.0568	0.0625	0.4741	0.4646	0.000300
4	-12	-8	DARA	DRRA	28	7.00	-88.0	0.1785	0.1665	-0.0615	0.1249	1.0575	0.1777	0.1650	-0.0685	0.1247	1.0625	0.0463	0.4764	0.4664	0.000224
4	-8	-4	DARA	DRRA	24	6.00	-40.0	0.1778	0.1653	-0.0625	0.1243	1.0610	0.1771	0.1640	-0.0689	0.1241	1.0653	0.0401	0.4778	0.4673	0.000185
4	-8	-4	DARA	DRRA	24	6.00	-44.0	0.1778	0.1653	-0.0625	0.1242	1.0610	0.1771	0.1640	-0.0689	0.1241	1.0653	0.0400	0.4778	0.4673	0.000185
4	-5	-1	DARA	DRRA	21	5.25	-10.0	0.1773	0.1645	-0.0632	0.1238	1.0634	0.1766	0.1633	-0.0692	0.1236	1.0672	0.0364	0.4790	0.4681	0.000155
4	-4	0	DARA	CRRA	20	5.00	0.0	0.1771	0.1642	-0.0634	0.1236	1.0641	0.1765	0.1630	-0.0693	0.1235	1.0679	0.0354	0.4794	0.4683	0.000145
4	-3	1	DARA	IRRA	19	4.75	8.0	0.1770	0.1639	-0.0636	0.1235	1.0648	0.1763	0.1628	-0.0693	0.1233	1.0685	0.0345	0.4798	0.4686	0.000134
4	-3	1	DARA	IRRA	19	4.75	8.5	0.1770	0.1639	-0.0636	0.1235	1.0648	0.1763	0.1628	-0.0693	0.1233	1.0685	0.0345	0.4798	0.4686	0.000135
4	-1	3	DARA	IRRA	17	4.25	24.0	0.1767	0.1634	-0.0640	0.1232	1.0662	0.1760	0.1623	-0.0695	0.1231	1.0698	0.0331	0.4807	0.4691	0.000113
4	0	4	CARA	IRRA	16	4.00	32.0	0.1765	0.1632	-0.0642	0.1230	1.0669	0.1759	0.1621	-0.0696	0.1229	1.0704	0.0326	0.4811	0.4694	0.000103

Table 10 provides the result from the optimization of portfolio allocation using OMX30 (large-capitalization Swedish stocks) excluding Nokia, from a sample of monthly observations between October 2003 and November 2016, plus a risk-free asset of 0.24% interest as the asset base. Each row corresponds to one specific investor defined by its risk aversion (RA), derivative of absolute risk aversion (dA/dW), derivative of relative risk aversion (dR/dW), skewness preference, skewness ratio, and derivative of skewness preference (dS/dW). For simplicity the type of risk aversion is included, with ARA meaning absolute risk aversion and RRA meaning relative risk aversion. These can be either increasing, constant or decreasing. For each investor there are two resulting portfolios, one when the investor considers mean, variance and skewness, and one when the investor only consider mean and variance. The portfolio moments and certainty equivalent are annualized. Euclidean distances are computed on the difference in the weights in assets between the MVS-optimized portfolio, the MV-optimized portfolio and the equally weighted portfolio. Difference in certainty equivalent (CE) is calculated as the CE of the MVS-portfolio less the CE of the MV-portfolio for each investor

5.4.3.2 Without risk-free

Figure 7 - Optimal portfolios in the MVS space for CARA/IRRA investor with RA=4

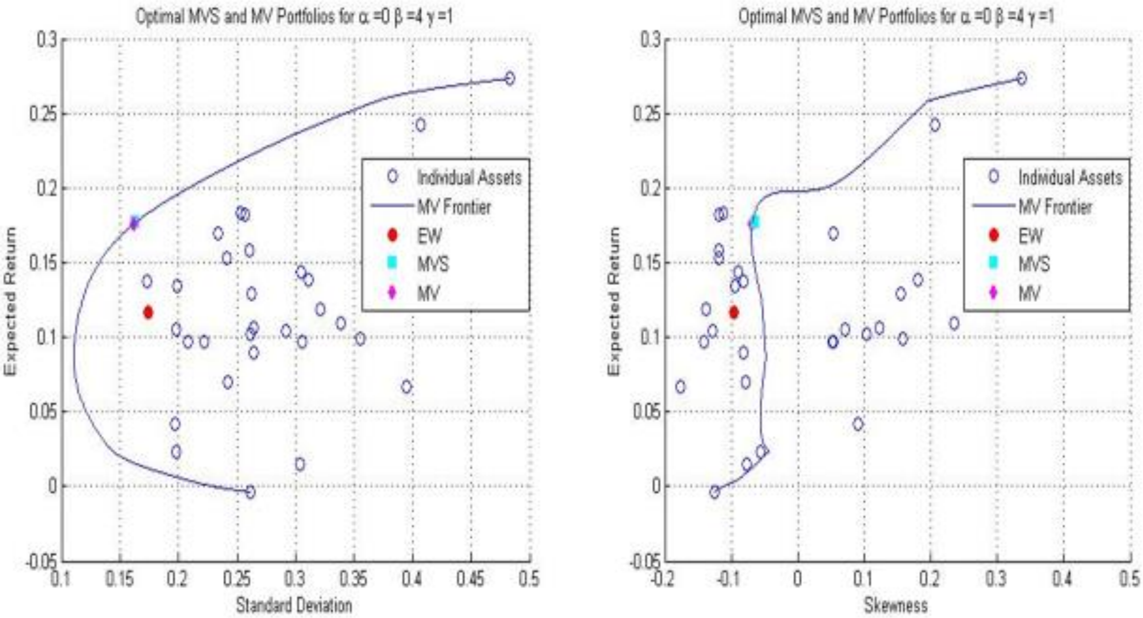

Figure 7 shows the results for a CARA/IRRA with RA=4 from the optimization of portfolio allocation using OMXS30 (large-capitalization Swedish stocks) excluding Nokia, from a sample of monthly observations between October 2003 and November 2016, not including a risk-free asset, as the asset base. MV represents the mean-variance-optimized portfolio and MVS the mean-variance-skewness-optimized portfolio. Each graph in the figure is presenting the same 3D-graph from two different perspectives, an expected return-standard deviation perspective and an expected return-skewness perspective. For comparison there is also the equally-weighted (EW) portfolio, the individual assets and the mean-variance efficient frontier.

Figure 8 - Optimal portfolios in the MVS space for DARA/DRRA investor with RA=16

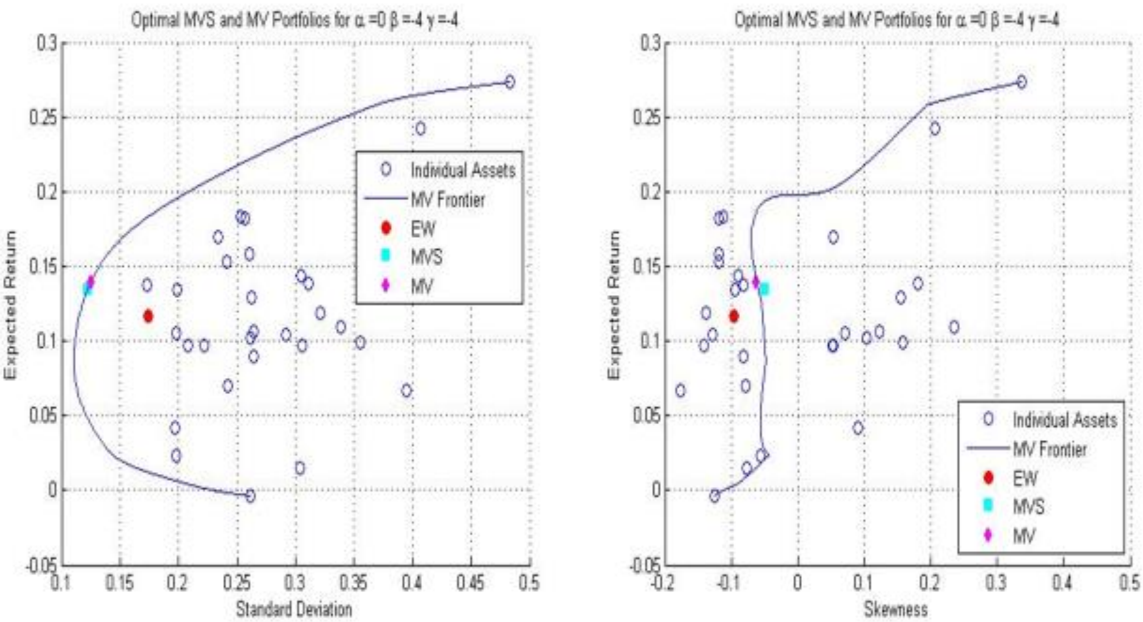

Figure 8 shows the results for a DARA/DRRA with RA=16 from the optimization of portfolio allocation using OMXS30 (large-capitalization Swedish stocks) excluding Nokia, from a sample of monthly observations between October 2003 and November 2016, not including a risk-free asset, as the asset base. MV represents the mean-variance-optimized portfolio and MVS the mean-variance-skewness-optimized portfolio. Each graph in the figure is presenting the same 3D-graph from two different perspectives, an expected return-standard deviation perspective and an expected return-skewness perspective. For comparison there is also the equally-weighted (EW) portfolio, the individual assets and the mean-variance efficient frontier.

Compared to the result with a risk-free asset, for the same assets and period of time, there is no decrease in CE differences and distances when we look at an RA=16 investor rather than an RA=8 investor. That also means that difference in CE is constantly increasing with skewness preference in this case. This provides some evidence that faults in skewness preference and ratio might not be the reason for the drop in the risk-free asset case, however not definitive evidence. Results are overall consistent with previous outcomes.

Table 8 - Swedish large cap optimal portfolios without a riskless asset (OMX 2003-16)

Arrow-Pratt Measures			Risk Type		Kane Measures			MVS-Optimized Portfolios				MV-Optimized Portfolios				Euclidean Distances			
RA	dA/dW	dR/dW	ARA	RRA	Skewness Preference	Skewness Ratio	dS/dW	Expected Return	Standard Deviation	Skewness	Certainty Equivalent	Expected Return	Standard Deviation	Skewness	Certainty Equivalent	MVS-MV	MVS-EW	MV-EW	Difference in CE
16	-80	-64	DARA	DRRA	336	21.00	-2368.0	0.1339	0.1229	-0.0491	0.0220	0.1400	0.1254	-0.0622	0.0191	0.0954	0.4414	0.4405	0.002859
16	-48	-32	DARA	DRRA	304	19.00	-1120.0	0.1331	0.1224	-0.0489	0.0202	0.1391	0.1248	-0.0618	0.0177	0.0947	0.4405	0.4393	0.002523
16	-32	-16	DARA	DRRA	288	18.00	-544.0	0.1327	0.1222	-0.0489	0.0193	0.1386	0.1245	-0.0616	0.0170	0.0944	0.4402	0.4388	0.002358
16	-32	-16	DARA	DRRA	288	18.00	-560.0	0.1327	0.1222	-0.0489	0.0193	0.1386	0.1245	-0.0616	0.0170	0.0944	0.4402	0.4388	0.002355
16	-20	-4	DARA	DRRA	276	17.25	-136.0	0.1324	0.1220	-0.0488	0.0187	0.1383	0.1243	-0.0615	0.0164	0.0942	0.4399	0.4384	0.002234
16	-16	0	DARA	CRRA	272	17.00	0.0	0.1323	0.1220	-0.0488	0.0185	0.1382	0.1242	-0.0614	0.0163	0.0941	0.4398	0.4383	0.002194
16	-12	4	DARA	IRRA	268	16.75	128.0	0.1322	0.1219	-0.0488	0.0182	0.1380	0.1241	-0.0614	0.0161	0.0941	0.4398	0.4382	0.002152
16	-12	4	DARA	IRRA	268	16.75	130.0	0.1322	0.1219	-0.0488	0.0182	0.1380	0.1241	-0.0614	0.0161	0.0941	0.4398	0.4382	0.002153
16	-4	12	DARA	IRRA	260	16.25	384.0	0.1321	0.1218	-0.0487	0.0178	0.1378	0.1240	-0.0613	0.0157	0.0940	0.4396	0.4380	0.002071
16	0	16	CARA	IRRA	256	16.00	512.0	0.1320	0.1218	-0.0487	0.0176	0.1377	0.1239	-0.0612	0.0156	0.0939	0.4395	0.4379	0.002031
8	-40	-32	DARA	DRRA	104	13.00	-672.0	0.1597	0.1421	-0.0606	0.0826	0.1593	0.1409	-0.0692	0.0818	0.0757	0.4984	0.4802	0.000837
8	-24	-16	DARA	DRRA	88	11.00	-304.0	0.1584	0.1406	-0.0619	0.0811	0.1585	0.1401	-0.0691	0.0805	0.0639	0.4934	0.4797	0.000622
8	-16	-8	DARA	DRRA	80	10.00	-144.0	0.1577	0.1398	-0.0624	0.0804	0.1581	0.1397	-0.0691	0.0799	0.0582	0.4901	0.4795	0.000530
8	-16	-8	DARA	DRRA	80	10.00	-152.0	0.1577	0.1398	-0.0624	0.0804	0.1581	0.1397	-0.0691	0.0799	0.0581	0.4901	0.4795	0.000528
8	-10	-2	DARA	DRRA	74	9.25	-36.0	0.1572	0.1392	-0.0627	0.0799	0.1578	0.1394	-0.0691	0.0794	0.0541	0.4878	0.4793	0.000467
8	-8	0	DARA	CRRA	72	9.00	0.0	0.1570	0.1390	-0.0628	0.0797	0.1577	0.1393	-0.0691	0.0793	0.0528	0.4871	0.4793	0.000448
8	-6	2	DARA	IRRA	70	8.75	32.0	0.1568	0.1389	-0.0630	0.0796	0.1576	0.1392	-0.0691	0.0791	0.0515	0.4863	0.4793	0.000429
8	-6	2	DARA	IRRA	70	8.75	33.0	0.1568	0.1389	-0.0630	0.0796	0.1576	0.1392	-0.0691	0.0791	0.0515	0.4863	0.4793	0.000429
8	-2	6	DARA	IRRA	66	8.25	96.0	0.1565	0.1385	-0.0632	0.0792	0.1574	0.1390	-0.0690	0.0788	0.0490	0.4848	0.4792	0.000394
8	0	8	CARA	IRRA	64	8.00	128.0	0.1563	0.1383	-0.0633	0.0791	0.1573	0.1390	-0.0690	0.0787	0.0478	0.4841	0.4792	0.000377
4	-20	-16	DARA	DRRA	36	9.00	-208.0	0.1801	0.1694	-0.0590	0.1262	0.1790	0.1671	-0.0676	0.1259	0.0625	0.4729	0.4634	0.000300
4	-12	-8	DARA	DRRA	28	7.00	-88.0	0.1785	0.1665	-0.0615	0.1249	0.1777	0.1650	-0.0685	0.1247	0.0463	0.4752	0.4651	0.000224
4	-8	-4	DARA	DRRA	24	6.00	-40.0	0.1778	0.1653	-0.0625	0.1243	0.1771	0.1640	-0.0689	0.1241	0.0401	0.4766	0.4661	0.000185
4	-8	-4	DARA	DRRA	24	6.00	-44.0	0.1778	0.1653	-0.0625	0.1242	0.1771	0.1640	-0.0689	0.1241	0.0400	0.4766	0.4661	0.000185
4	-5	-1	DARA	DRRA	21	5.25	-10.0	0.1773	0.1645	-0.0632	0.1238	0.1766	0.1633	-0.0692	0.1236	0.0364	0.4778	0.4668	0.000155
4	-4	0	DARA	CRRA	20	5.00	0.0	0.1771	0.1642	-0.0634	0.1236	0.1765	0.1630	-0.0693	0.1235	0.0354	0.4782	0.4671	0.000145
4	-3	1	DARA	IRRA	19	4.75	8.0	0.1770	0.1639	-0.0636	0.1235	0.1763	0.1628	-0.0693	0.1233	0.0345	0.4786	0.4673	0.000134
4	-3	1	DARA	IRRA	19	4.75	8.5	0.1770	0.1639	-0.0636	0.1235	0.1763	0.1628	-0.0693	0.1233	0.0345	0.4786	0.4673	0.000135
4	-1	3	DARA	IRRA	17	4.25	24.0	0.1767	0.1634	-0.0640	0.1232	0.1760	0.1623	-0.0695	0.1231	0.0331	0.4795	0.4679	0.000113
4	0	4	CARA	IRRA	16	4.00	32.0	0.1765	0.1632	-0.0642	0.1230	0.1759	0.1621	-0.0696	0.1229	0.0326	0.4799	0.4681	0.000103

The table provides the result from the optimization of portfolio allocation using OMX30 (large-capitalization Swedish stocks) excluding Nokia, from a sample of monthly observations between October 2003 and November 2016, not including a risk-free asset, as the asset base. Each row corresponds to one specific investor defined by its risk aversion (RA), derivative of absolute risk aversion (dA/dW), derivative of relative risk aversion (dR/dW), skewness preference, skewness ratio, and derivative of skewness preference (dS/dW). For simplicity the type of risk aversion is included, with ARA meaning absolute risk aversion and RRA meaning relative risk aversion. These can be either increasing, constant or decreasing. For each investor there are two resulting portfolios, one when the investor considers mean, variance and skewness, and one when the investor only consider mean and variance. The portfolio moments and certainty equivalent are annualized. Euclidean distances are computed on the difference in the weights in assets between the MVS-optimized portfolio, the MV-optimized portfolio and the equally weighted portfolio. Difference in certainty equivalent (CE) is calculated as the CE of the MVS-portfolio less the CE of the MV-portfolio for each investor.

5.4.4 DAX30 2011-2016

Figure 9 - Optimal portfolios in the MVS space for CARA/IRRA investor with RA=4

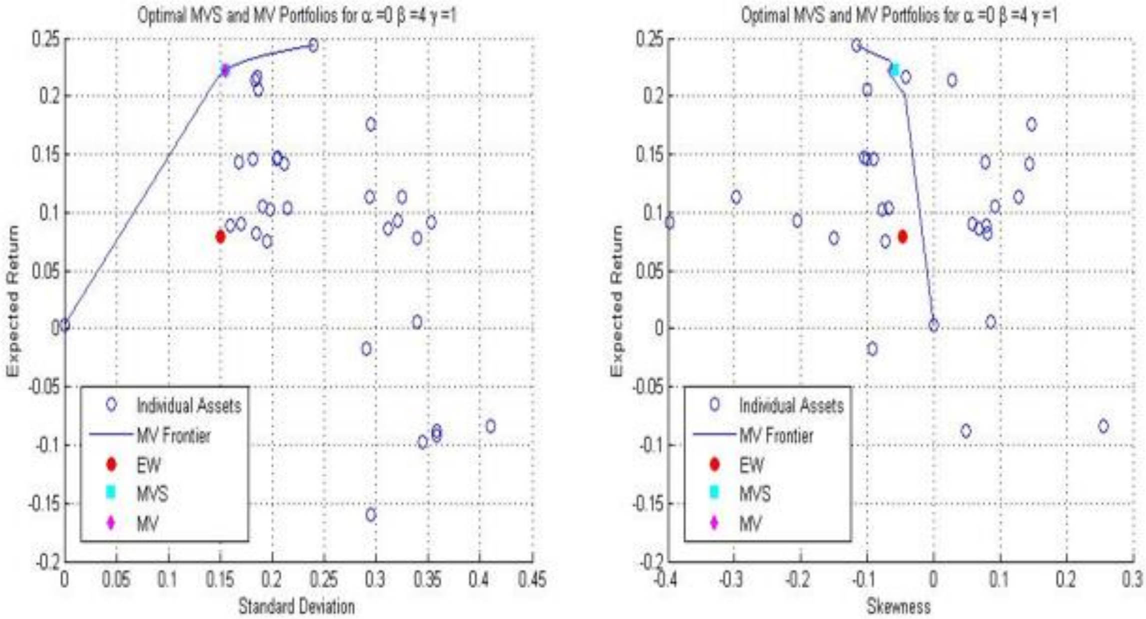

Figure 9 shows the results for a CARA/IRRA with RA=4 from the optimization of portfolio allocation using DAX30 (large-capitalization German stocks), from a sample of monthly observations between October 2011 and November 2016, plus a risk-free asset of 0.27% interest as the asset base. MV represents the mean-variance-optimized portfolio and MVS the mean-variance-skewness-optimized portfolio. Each graph in the figure is presenting the same 3D-graph from two different perspectives, an expected return-standard deviation perspective and an expected return-skewness perspective. For comparison there is also the equally-weighted (EW) portfolio, the individual assets and the mean-variance efficient frontier.

Figure 10 - Optimal portfolios in the MVS space for DARA/DRRA investor with RA=16

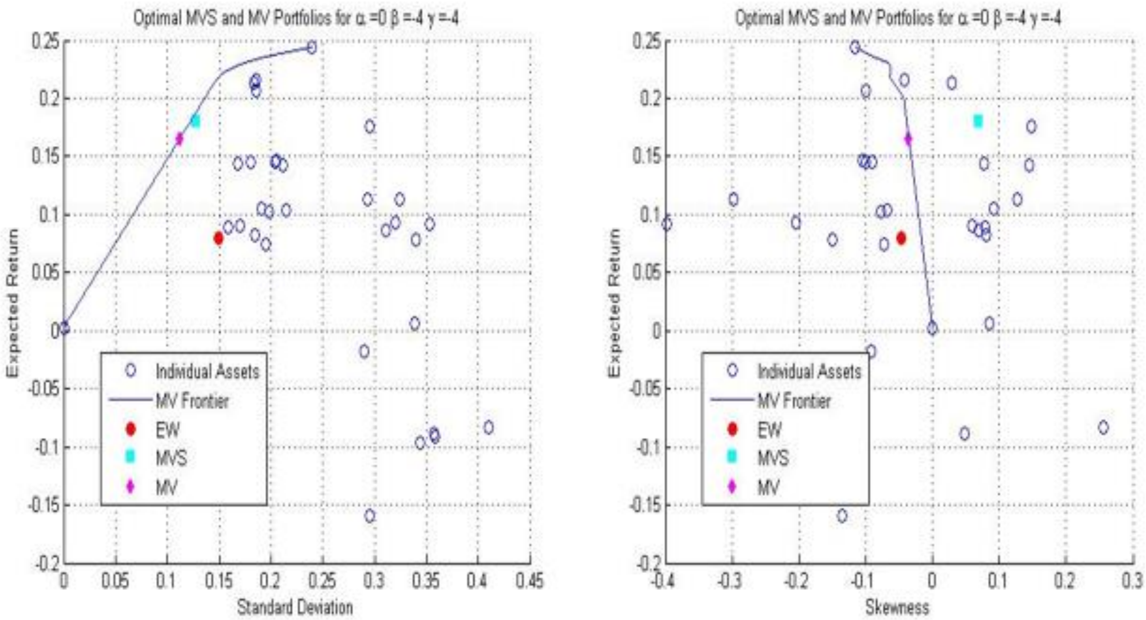

Figure 10 shows the results for a DARA/DRRA with RA=16 from the optimization of portfolio allocation using DAX30 (large-capitalization German stocks), from a sample of monthly observations between October 2011 and November 2016, plus a risk-free asset of 0.27% interest as the asset base. MV represents the mean-variance-optimized portfolio and MVS the mean-variance-skewness-optimized portfolio. Each graph in the figure is presenting the same 3D-graph from two different perspectives, an expected return-standard deviation perspective and an expected return-skewness perspective. For comparison there is also the equally-weighted (EW) portfolio, the individual assets and the mean-variance efficient frontier.

There are barely any differences in these results as compared to OMX30 from the same period of time. The expected return, standard deviation, CE and difference in CE are all slightly lower. The skewness is mostly negative here, except for the MVS-investors with RA=16. One interesting difference is that skewness becomes more negative for MVS-investors with RA=8 when moving from IRRA to DRRA investors even though MV-investors behave the opposite. Our best guess is that this is asset specific but we don’t have any specific evidence that contradicts or support it. However, skewness is moving “correctly” when investors have RA=16. Overall these results just add some robustness to the previous tables.

Table 8 - German large cap optimal portfolios with a riskless asset (DAX 2011-16)

Arrow-Pratt Measures			Risk Type		Kane Measures			MVS-Optimized Portfolios					MV-Optimized Portfolios					Euclidean Distances			
RA	dA/dW	dR/dW	ARA	RRA	Skewness Preference	Skewness Ratio	dS/dW	Expected Return	Standard Deviation	Skewness	Certainty Equivalent	Sharpe Ratio	Expected Return	Standard Deviation	Skewness	Certainty Equivalent	Sharpe Ratio	MVS-MV	MVS-EW	MV-EW	Difference in CE
16	-80	-64	DARA	DRRA	336	21.00	-2368.0	0.1794	0.1276	0.0695	0.0812	1.3847	0.1646	0.1119	-0.0354	0.0743	1.4477	0.3165	0.4950	0.3910	0.006929
16	-48	-32	DARA	DRRA	304	19.00	-1120.0	0.1790	0.1271	0.0688	0.0775	1.3870	0.1571	0.1067	-0.0338	0.0726	1.4477	0.3416	0.4920	0.3932	0.004902
16	-32	-16	DARA	DRRA	288	18.00	-544.0	0.1739	0.1230	0.0654	0.0756	1.3923	0.1538	0.1044	-0.0330	0.0718	1.4477	0.3177	0.4718	0.3956	0.003854
16	-32	-16	DARA	DRRA	288	18.00	-560.0	0.1738	0.1229	0.0653	0.0756	1.3924	0.1538	0.1043	-0.0330	0.0717	1.4477	0.3168	0.4713	0.3957	0.003841
16	-20	-4	DARA	DRRA	276	17.25	-136.0	0.1670	0.1174	0.0607	0.0744	1.3989	0.1514	0.1027	-0.0325	0.0712	1.4477	0.2770	0.4493	0.3979	0.003209
16	-16	0	DARA	CRRA	272	17.00	0.0	0.1650	0.1159	0.0594	0.0740	1.4007	0.1506	0.1022	-0.0323	0.0710	1.4477	0.2661	0.4440	0.3987	0.003023
16	-12	4	DARA	IRRA	268	16.75	128.0	0.1631	0.1144	0.0582	0.0736	1.4025	0.1499	0.1017	-0.0322	0.0708	1.4477	0.2558	0.4394	0.3995	0.002845
16	-12	4	DARA	IRRA	268	16.75	130.0	0.1631	0.1144	0.0582	0.0736	1.4025	0.1499	0.1017	-0.0322	0.0708	1.4477	0.2559	0.4395	0.3995	0.002846
16	-4	12	DARA	IRRA	260	16.25	384.0	0.1594	0.1114	0.0552	0.0729	1.4073	0.1484	0.1007	-0.0319	0.0704	1.4477	0.2313	0.4291	0.4013	0.002523
16	0	16	CARA	IRRA	256	16.00	512.0	0.1576	0.1099	0.0535	0.0726	1.4102	0.1477	0.1002	-0.0317	0.0702	1.4477	0.2177	0.4238	0.4022	0.002381
8	-40	-32	DARA	DRRA	104	13.00	-672.0	0.2174	0.1505	-0.0500	0.1349	1.4269	0.2196	0.1514	-0.0645	0.1340	1.4319	0.1049	0.5156	0.4955	0.000826
8	-24	-16	DARA	DRRA	88	11.00	-304.0	0.2155	0.1487	-0.0498	0.1323	1.4315	0.2194	0.1513	-0.0647	0.1316	1.4323	0.0901	0.5022	0.4951	0.000729
8	-16	-8	DARA	DRRA	80	10.00	-144.0	0.2147	0.1479	-0.0498	0.1310	1.4335	0.2193	0.1512	-0.0648	0.1303	1.4325	0.0846	0.4963	0.4950	0.000701
8	-16	-8	DARA	DRRA	80	10.00	-152.0	0.2147	0.1479	-0.0498	0.1310	1.4336	0.2193	0.1512	-0.0648	0.1303	1.4325	0.0845	0.4962	0.4950	0.000699
8	-10	-2	DARA	DRRA	74	9.25	-36.0	0.2141	0.1473	-0.0498	0.1301	1.4349	0.2192	0.1511	-0.0648	0.1294	1.4326	0.0814	0.4921	0.4949	0.000688
8	-8	0	DARA	CRRA	72	9.00	0.0	0.2139	0.1471	-0.0498	0.1297	1.4354	0.2192	0.1511	-0.0649	0.1291	1.4327	0.0805	0.4907	0.4948	0.000685
8	-6	2	DARA	IRRA	70	8.75	32.0	0.2137	0.1469	-0.0499	0.1294	1.4358	0.2192	0.1511	-0.0649	0.1287	1.4327	0.0797	0.4894	0.4948	0.000683
8	-6	2	DARA	IRRA	70	8.75	33.0	0.2137	0.1469	-0.0499	0.1294	1.4358	0.2192	0.1511	-0.0649	0.1287	1.4327	0.0797	0.4894	0.4948	0.000684
8	-2	6	DARA	IRRA	66	8.25	96.0	0.2133	0.1466	-0.0499	0.1288	1.4367	0.2191	0.1511	-0.0649	0.1281	1.4328	0.0783	0.4868	0.4948	0.000682
8	0	8	CARA	IRRA	64	8.00	128.0	0.2131	0.1464	-0.0499	0.1285	1.4371	0.2191	0.1511	-0.0649	0.1278	1.4328	0.0778	0.4856	0.4947	0.000683
4	-20	-16	DARA	DRRA	36	9.00	-208.0	0.2234	0.1573	-0.0531	0.1778	1.4026	0.2232	0.1565	-0.0626	0.1775	1.4087	0.0878	0.5478	0.5210	0.000249
4	-12	-8	DARA	DRRA	28	7.00	-88.0	0.2230	0.1565	-0.0556	0.1763	1.4077	0.2228	0.1559	-0.0626	0.1761	1.4118	0.0688	0.5369	0.5174	0.000157
4	-8	-4	DARA	DRRA	24	6.00	-40.0	0.2228	0.1561	-0.0567	0.1755	1.4100	0.2227	0.1557	-0.0627	0.1754	1.4132	0.0594	0.5319	0.5158	0.000119
4	-8	-4	DARA	DRRA	24	6.00	-44.0	0.2228	0.1561	-0.0568	0.1755	1.4101	0.2227	0.1557	-0.0627	0.1754	1.4132	0.0592	0.5319	0.5157	0.000118
4	-5	-1	DARA	DRRA	21	5.25	-10.0	0.2227	0.1559	-0.0576	0.1750	1.4116	0.2226	0.1555	-0.0627	0.1749	1.4142	0.0524	0.5284	0.5146	0.000093
4	-4	0	DARA	CRRA	20	5.00	0.0	0.2227	0.1558	-0.0578	0.1748	1.4121	0.2225	0.1554	-0.0627	0.1747	1.4145	0.0500	0.5273	0.5142	0.000085
4	-3	1	DARA	IRRA	19	4.75	8.0	0.2226	0.1557	-0.0581	0.1746	1.4127	0.2225	0.1554	-0.0627	0.1745	1.4149	0.0477	0.5261	0.5138	0.000078
4	-3	1	DARA	IRRA	19	4.75	8.5	0.2226	0.1557	-0.0581	0.1746	1.4127	0.2225	0.1554	-0.0627	0.1745	1.4149	0.0477	0.5261	0.5138	0.000078
4	-1	3	DARA	IRRA	17	4.25	24.0	0.2225	0.1555	-0.0586	0.1742	1.4136	0.2224	0.1552	-0.0628	0.1741	1.4155	0.0430	0.5239	0.5131	0.000064
4	0	4	CARA	IRRA	16	4.00	32.0	0.2225	0.1554	-0.0589	0.1740	1.4141	0.2224	0.1552	-0.0628	0.1740	1.4158	0.0407	0.5229	0.5127	0.000057

The table provides the result from the optimization of portfolio allocation using DAX30 (large-capitalization German stocks), from a sample of monthly observations between October 2011 and November 2016, plus a risk-free asset of 0.27% interest as the asset base. Each row corresponds to one specific investor defined by its risk aversion (RA), derivative of absolute risk aversion (dA/dW), derivative of relative risk aversion (dR/dW), skewness preference, skewness ratio, and derivative of skewness preference (dS/dW). For simplicity the type of risk aversion is included, with ARA meaning absolute risk aversion and RRA meaning relative risk aversion. These can be either increasing, constant or decreasing. For each investor there are two resulting portfolios, one when the investor considers mean, variance and skewness, and one when the investor only consider mean and variance. The portfolio moments and certainty equivalent are annualized. Euclidean distances are computed on the difference in the weights in assets between the MVS-optimized portfolio, the MV-optimized portfolio and the equally weighted portfolio. Difference in certainty equivalent (CE) is calculated as the CE of the MVS-portfolio less the CE of the MV-portfolio for each investor.

5.5 General patterns and considerations

In general, the difference in certain equivalent is increasing with skewness preference and ratio, when investors become more risk averse, and when moving from the IRRA to the DRRA end of the spectrum. However, since the pattern is not perfect between the samples, it seems like skewness preference and skewness ratio is missing some information about how the pattern of differences in CE will be moving, but there is some possibility that it could be because of the assets. On the other hand, the pattern of distance between MV and MVS portfolios measured by the Euclidean norm appears to be more asset specific and its pattern is not the same as for the difference in CE. Additionally, we noticed that the results for differences in CE and distance are not only asset specific but also can vary a lot dependent on the time period that is being used as input. Anyhow, we found economically significant differences for some investors and they should consider skewness when investing. To determine whether or not an investor should consider skewness one can look into her skewness preference and skewness ratio, the higher, the more likely the investor will get significant differences in their portfolio allocation when adding skewness.

Further, for the pairwise investors, as far as we saw, the difference in CE and distance was always higher for the investor with more positive or less negative derivative of skewness preference but these differences in differences was very small and don't seem to be of any importance. However, if you would have a long one-period investment this derivative might have some effect since wealth would change substantial and thus the skewness preference, even though not very likely.

Expected return and variance is naturally decreasing with risk aversion but increasing when moving from IRRA to DRRA investors. Skewness is mostly moving towards positive numbers for MVS-investors when being more risk averse, but it is somewhat asset specific since there is one case where this is not true and since MV-investors often move in the same direction. Also, skewness is mostly increasing when moving from IRRA to DRRA-investors, although there are cases as we have seen that shows that also this is somewhat asset specific. However, skewness is always more positive or less negative for MVS-investors and when skewness is getting more positive for MV-investors it is generally getting even more positive for MVS-investors.

Moreover, the difference in Sharpe ratio between MV and MVS-investors, when applicable, shows that MVS-investors are investing in a suboptimal way according to a mean-variance framework. The differences that we noticed also means that the separation theorem doesn't hold in a MVS framework and that there are different tangency portfolios dependent on the investors preference and not just one, contradicting fundamental financial theory.

Lastly, we might have found a reason why someone rationally would be under diversified by holding single stocks in too high proportion. This is because diversified asset such as indexes or diversified mutual funds seems to exhibit less possibilities to increase positive skewness than individual assets, as supported from previous research.

6 Conclusion

The mean-variance analysis proposed by Markowitz can be considered the backbone of modern financial theory and plays a central role both in literature and in practice. There has however been a lot of scepticism over the accuracy of the mean-variance method, due to its strong, over simplistic assumption of normality of returns or quadratic utility. Several authors have provided evidence against these assumptions and incorporated skewness in the picture. The question whether to include skewness and higher order moments in portfolio allocation however, has got diverse answers. In this paper we have shed some light to some of the previous findings as well as investigated whether the type of risk aversion the investor has matters when considering to include skewness into the portfolio allocation problem. We did this by using a Taylor series of a general utility function, that we developed ourselves. Using this we optimize the portfolio allocation for different investors using both a mean-variance framework and a mean-variance-skewness framework. Our findings are in line with previous literature, variation in underlying investable assets seem to matter for the result. Also in line with previous research, we find that more risk averse investors care more about skewness and thus are worse off when the skewness dimension is not incorporated in the portfolio allocation decision. Further, we also find that investors with larger decreasing absolute risk aversion (DARA) care more about skewness compared to other investors in the other side of the risk aversion spectrum. The differences are for some investors economically significant and it is important that these investors consider skewness when doing their portfolio allocation. We found that one way to tell if investors should include skewness is to investigate their skewness preference and skewness ratio, the higher, the more likely skewness makes a difference for the investor.

The work by Sharpe, efficient portfolios having maximum Sharpe ratio, and Tobin's separation theorem between one common risky portfolio and cash, are challenged by our framework. We found that neither of these holds when skewness is included in the picture. In the results, we saw that Sharpe ratio is inferior for investors who consider skewness and that Sharpe ratio varies between those investors, which means that in the MVS case there is no common tangency portfolio. In line with this we also find rational reason why investors might invest relatively large part of their wealth in individual stocks. This is because that most of the opportunity to increase positive skewness is diversified away in well diversified assets such as indexes, which also other authors have claimed.

It is however important to remember that several authors have raised concerns about the use of a Taylor series since it might not converge, as well as badly approximate the true utility if truncated to early, for example when the distribution of returns have a too high skewness. We do not think it is a problem with the results that we have reported.

As an additional remark, we should outline that since some investors seem to care about skewness, it is possible that they also display specific preferences about kurtosis, which could be looked into in future research. Additionally, it is also possible to extend our utility function with extra parameters to observe behaviours from a wider set of different investors. Finally, even if widely used in theory, there are some doubts whether expected utility is a good approximation of reality. Some have already researched the area of portfolio allocation within cumulative prospect theory and some of our work could be extended into this specific area.

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Appendix 1

MSCI optimal portfolios without a riskless asset

Arrow-Pratt Measures			Risk Type		Kane Measures			MVS-Optimized Portfolios				MV-Optimized Portfolios				Euclidean Distances			
RA	dA/dW	dR/dW	ARA	RRA	Skewness Preference	Skewness Ratio	dS/dW	Expected Return	Standard Deviation	Skewness	Certainty Equivalent	Expected Return	Standard Deviation	Skewness	Certainty Equivalent	MVS-MV	MVS-EW	MV-EW	Difference in CE
16	-80	-64	DARA	DRRA	336	21.00	-2368.0	0.1676	0.1065	0.0416	0.0903	0.1610	0.1010	0.0266	0.0890	0.1542	0.5347	0.4699	0.001348
16	-48	-32	DARA	DRRA	304	19.00	-1120.0	0.1667	0.1059	0.0413	0.0881	0.1599	0.1002	0.0248	0.0870	0.1681	0.5335	0.4657	0.001111
16	-32	-16	DARA	DRRA	288	18.00	-544.0	0.1663	0.1056	0.0412	0.0870	0.1594	0.0998	0.0239	0.0860	0.1749	0.5331	0.4639	0.000974
16	-32	-16	DARA	DRRA	288	18.00	-560.0	0.1663	0.1056	0.0412	0.0870	0.1594	0.0998	0.0239	0.0860	0.1750	0.5331	0.4639	0.000972
16	-20	-4	DARA	DRRA	276	17.25	-136.0	0.1659	0.1053	0.0409	0.0862	0.1591	0.0996	0.0232	0.0853	0.1780	0.5318	0.4626	0.000863
16	-16	0	DARA	CRRA	272	17.00	0.0	0.1656	0.1050	0.0403	0.0859	0.1589	0.0995	0.0230	0.0850	0.1726	0.5281	0.4622	0.000827
16	-12	4	DARA	IRRA	268	16.75	128.0	0.1652	0.1046	0.0398	0.0856	0.1588	0.0994	0.0228	0.0848	0.1674	0.5245	0.4618	0.000792
16	-12	4	DARA	IRRA	268	16.75	130.0	0.1652	0.1046	0.0398	0.0856	0.1588	0.0994	0.0228	0.0848	0.1675	0.5245	0.4618	0.000793
16	-4	12	DARA	IRRA	260	16.25	384.0	0.1645	0.1040	0.0387	0.0851	0.1586	0.0992	0.0223	0.0843	0.1577	0.5180	0.4611	0.000728
16	0	16	CARA	IRRA	256	16.00	512.0	0.1642	0.1037	0.0382	0.0848	0.1584	0.0991	0.0221	0.0841	0.1532	0.5150	0.4607	0.000698
8	-40	-32	DARA	DRRA	104	13.00	-672.0	0.1850	0.1246	0.0528	0.1307	0.1817	0.1204	0.0494	0.1305	0.0681	0.6275	0.5988	0.000170
8	-24	-16	DARA	DRRA	88	11.00	-304.0	0.1830	0.1220	0.0510	0.1288	0.1805	0.1190	0.0485	0.1287	0.0519	0.6102	0.5900	0.000107
8	-16	-8	DARA	DRRA	80	10.00	-144.0	0.1821	0.1209	0.0502	0.1279	0.1799	0.1183	0.0481	0.1278	0.0450	0.6028	0.5859	0.000084
8	-16	-8	DARA	DRRA	80	10.00	-152.0	0.1820	0.1209	0.0502	0.1279	0.1799	0.1183	0.0481	0.1278	0.0449	0.6027	0.5859	0.000083
8	-10	-2	DARA	DRRA	74	9.25	-36.0	0.1814	0.1201	0.0496	0.1272	0.1794	0.1178	0.0478	0.1271	0.0402	0.5976	0.5830	0.000069
8	-8	0	DARA	CRRA	72	9.00	0.0	0.1812	0.1199	0.0495	0.1269	0.1793	0.1177	0.0477	0.1269	0.0387	0.5960	0.5821	0.000064
8	-6	2	DARA	IRRA	70	8.75	32.0	0.1810	0.1196	0.0493	0.1267	0.1791	0.1175	0.0476	0.1266	0.0372	0.5944	0.5811	0.000060
8	-6	2	DARA	IRRA	70	8.75	33.0	0.1810	0.1196	0.0493	0.1267	0.1791	0.1175	0.0476	0.1266	0.0372	0.5944	0.5811	0.000060
8	-2	6	DARA	IRRA	66	8.25	96.0	0.1805	0.1191	0.0489	0.1263	0.1789	0.1172	0.0474	0.1262	0.0344	0.5913	0.5793	0.000052
8	0	8	CARA	IRRA	64	8.00	128.0	0.1803	0.1189	0.0488	0.1260	0.1787	0.1171	0.0473	0.1260	0.0330	0.5898	0.5784	0.000049
4	-20	-16	DARA	DRRA	36	9.00	-208.0	0.1927	0.1358	0.0628	0.1601	0.1919	0.1341	0.0604	0.1600	0.0641	0.7235	0.7020	0.000067
4	-12	-8	DARA	DRRA	28	7.00	-88.0	0.1923	0.1350	0.0618	0.1588	0.1918	0.1338	0.0600	0.1587	0.0477	0.7136	0.6988	0.000039
4	-8	-4	DARA	DRRA	24	6.00	-40.0	0.1922	0.1347	0.0613	0.1581	0.1917	0.1337	0.0598	0.1581	0.0402	0.7093	0.6972	0.000029
4	-8	-4	DARA	DRRA	24	6.00	-44.0	0.1922	0.1347	0.0613	0.1581	0.1917	0.1337	0.0598	0.1581	0.0400	0.7092	0.6972	0.000029
4	-5	-1	DARA	DRRA	21	5.25	-10.0	0.1921	0.1344	0.0609	0.1576	0.1917	0.1336	0.0597	0.1576	0.0347	0.7062	0.6961	0.000022
4	-4	0	DARA	CRRA	20	5.00	0.0	0.1921	0.1344	0.0608	0.1574	0.1917	0.1336	0.0596	0.1574	0.0330	0.7053	0.6958	0.000020
4	-3	1	DARA	IRRA	19	4.75	8.0	0.1920	0.1343	0.0607	0.1573	0.1917	0.1336	0.0596	0.1572	0.0312	0.7043	0.6954	0.000018
4	-3	1	DARA	IRRA	19	4.75	8.5	0.1920	0.1343	0.0607	0.1573	0.1917	0.1336	0.0596	0.1572	0.0312	0.7043	0.6954	0.000018
4	-1	3	DARA	IRRA	17	4.25	24.0	0.1919	0.1341	0.0605	0.1569	0.1916	0.1335	0.0595	0.1569	0.0278	0.7025	0.6947	0.000015
4	0	4	CARA	IRRA	16	4.00	32.0	0.1919	0.1341	0.0604	0.1568	0.1916	0.1335	0.0594	0.1567	0.0261	0.7016	0.6944	0.000013

Appendix 1 provides the result from the optimization of portfolio allocation using 20 MSCI indexes, from a sample of monthly observations between October 2011 and August 2016, not including a risk-free asset, as the asset base. Each row corresponds to one specific investor defined by its risk aversion (RA), derivative of absolute risk aversion (dA/dW), derivative of relative risk aversion (dR/dW), skewness preference, skewness ratio, and derivative of skewness preference (dS/dW). For simplicity the type of risk aversion is included, with ARA meaning absolute risk aversion and RRA meaning relative risk aversion. These can be either increasing, constant or decreasing. For each investor there are two resulting portfolios, one when the investor considers mean, variance and skewness, and one when the investor only consider mean and variance. The portfolio moments and certainty equivalent are annualized. Euclidean distances are computed on the difference in the weights in assets between the MVS-optimized portfolio, the MV-optimized portfolio and the equally weighted portfolio. Difference in certainty equivalent (CE) is calculated as the CE of the MVS-portfolio less the CE of the MV-portfolio for each investor.

Appendix 2

Swedish large cap optimal portfolios without a riskless asset

Arrow-Pratt Measures			Risk Type		Kane Measures			MVS-Optimized Portfolios				MV-Optimized Portfolios				Euclidean Distances			
RA	dA/dW	dR/dW	ARA	RRA	Skewness Preference	Skewness Ratio	dS/dW	Expected Return	Standard Deviation	Skewness	Certainty Equivalent	Expected Return	Standard Deviation	Skewness	Certainty Equivalent	MVS-MV	MVS-EW	MV-EW	Difference in CE
16	-80	-64	DARA	DRRA	336	21.00	-2368.0	0.2142	0.1469	0.0831	0.0932	0.2091	0.1398	0.0606	0.0862	0.2369	0.5361	0.4622	0.006994
16	-48	-32	DARA	DRRA	304	19.00	-1120.0	0.2130	0.1455	0.0815	0.0878	0.2066	0.1383	0.0592	0.0813	0.2178	0.5228	0.4540	0.006430
16	-32	-16	DARA	DRRA	288	18.00	-544.0	0.2125	0.1448	0.0806	0.0850	0.2055	0.1377	0.0586	0.0789	0.2089	0.5166	0.4502	0.006110
16	-32	-16	DARA	DRRA	288	18.00	-560.0	0.2124	0.1448	0.0806	0.0850	0.2054	0.1377	0.0586	0.0789	0.2087	0.5165	0.4501	0.006101
16	-20	-4	DARA	DRRA	276	17.25	-136.0	0.2120	0.1444	0.0800	0.0830	0.2045	0.1371	0.0581	0.0771	0.2025	0.5121	0.4469	0.005859
16	-16	0	DARA	CRRA	272	17.00	0.0	0.2118	0.1442	0.0798	0.0823	0.2042	0.1369	0.0579	0.0765	0.2005	0.5106	0.4457	0.005775
16	-12	4	DARA	IRRA	268	16.75	128.0	0.2117	0.1441	0.0796	0.0816	0.2038	0.1367	0.0578	0.0759	0.1985	0.5092	0.4445	0.005686
16	-12	4	DARA	IRRA	268	16.75	130.0	0.2117	0.1441	0.0796	0.0816	0.2038	0.1367	0.0578	0.0759	0.1985	0.5092	0.4445	0.005687
16	-4	12	DARA	IRRA	260	16.25	384.0	0.2114	0.1438	0.0792	0.0802	0.2031	0.1363	0.0574	0.0747	0.1945	0.5063	0.4418	0.005528
16	0	16	CARA	IRRA	256	16.00	512.0	0.2112	0.1436	0.0790	0.0796	0.2027	0.1361	0.0572	0.0741	0.1927	0.5049	0.4403	0.005454
8	-40	-32	DARA	DRRA	104	13.00	-672.0	0.2262	0.1532	0.0774	0.1494	0.2261	0.1521	0.0684	0.1484	0.1248	0.5896	0.5872	0.000989
8	-24	-16	DARA	DRRA	88	11.00	-304.0	0.2256	0.1523	0.0761	0.1458	0.2254	0.1514	0.0683	0.1450	0.1037	0.5803	0.5777	0.000776
8	-16	-8	DARA	DRRA	80	10.00	-144.0	0.2253	0.1519	0.0754	0.1439	0.2251	0.1511	0.0682	0.1432	0.0939	0.5761	0.5731	0.000678
8	-16	-8	DARA	DRRA	80	10.00	-152.0	0.2253	0.1519	0.0754	0.1439	0.2251	0.1511	0.0682	0.1432	0.0937	0.5760	0.5731	0.000676
8	-10	-2	DARA	DRRA	74	9.25	-36.0	0.2250	0.1516	0.0749	0.1425	0.2248	0.1509	0.0681	0.1419	0.0869	0.5730	0.5698	0.000607
8	-8	0	DARA	CRRA	72	9.00	0.0	0.2250	0.1515	0.0748	0.1420	0.2247	0.1508	0.0681	0.1414	0.0846	0.5720	0.5687	0.000585
8	-6	2	DARA	IRRA	70	8.75	32.0	0.2249	0.1514	0.0746	0.1415	0.2247	0.1507	0.0681	0.1410	0.0824	0.5710	0.5676	0.000563
8	-6	2	DARA	IRRA	70	8.75	33.0	0.2249	0.1514	0.0746	0.1415	0.2247	0.1507	0.0681	0.1410	0.0824	0.5710	0.5676	0.000563
8	-2	6	DARA	IRRA	66	8.25	96.0	0.2248	0.1512	0.0743	0.1406	0.2245	0.1506	0.0680	0.1400	0.0780	0.5691	0.5655	0.000519
8	0	8	CARA	IRRA	64	8.00	128.0	0.2247	0.1511	0.0742	0.1401	0.2244	0.1505	0.0680	0.1396	0.0759	0.5682	0.5644	0.000498
4	-20	-16	DARA	DRRA	36	9.00	-208.0	0.2359	0.1651	0.0734	0.1888	0.2368	0.1664	0.0664	0.1885	0.0883	0.7235	0.7443	0.000265
4	-12	-8	DARA	DRRA	28	7.00	-88.0	0.2356	0.1646	0.0726	0.1865	0.2360	0.1649	0.0668	0.1863	0.0737	0.7195	0.7304	0.000169
4	-8	-4	DARA	DRRA	24	6.00	-40.0	0.2352	0.1639	0.0718	0.1853	0.2355	0.1642	0.0669	0.1852	0.0626	0.7141	0.7238	0.000126
4	-8	-4	DARA	DRRA	24	6.00	-44.0	0.2352	0.1639	0.0718	0.1853	0.2355	0.1642	0.0669	0.1852	0.0624	0.7141	0.7238	0.000125
4	-5	-1	DARA	DRRA	21	5.25	-10.0	0.2350	0.1634	0.0713	0.1845	0.2352	0.1637	0.0670	0.1844	0.0546	0.7103	0.7190	0.000098
4	-4	0	DARA	CRRA	20	5.00	0.0	0.2349	0.1633	0.0711	0.1842	0.2351	0.1635	0.0671	0.1841	0.0520	0.7091	0.7175	0.000089
4	-3	1	DARA	IRRA	19	4.75	8.0	0.2348	0.1631	0.0710	0.1839	0.2350	0.1633	0.0671	0.1838	0.0494	0.7079	0.7159	0.000081
4	-3	1	DARA	IRRA	19	4.75	8.5	0.2348	0.1631	0.0710	0.1839	0.2350	0.1633	0.0671	0.1838	0.0494	0.7079	0.7159	0.000081
4	-1	3	DARA	IRRA	17	4.25	24.0	0.2346	0.1628	0.0706	0.1833	0.2348	0.1630	0.0672	0.1832	0.0442	0.7055	0.7128	0.000066
4	0	4	CARA	IRRA	16	4.00	32.0	0.2345	0.1627	0.0705	0.1830	0.2347	0.1629	0.0672	0.1829	0.0417	0.7043	0.7113	0.000059

The table provides the result from the optimization of portfolio allocation using OMXS30 (large-capitalization Swedish stocks), from a sample of monthly observations between October 2011 and November 2016, not including a risk-free asset, as the asset base. Each row corresponds to one specific investor defined by its risk aversion (RA), derivative of absolute risk aversion (dA/dW), derivative of relative risk aversion (dR/dW), skewness preference, skewness ratio, and derivative of skewness preference (dS/dW). For simplicity the type of risk aversion is included, with ARA meaning absolute risk aversion and RRA meaning relative risk aversion. These can be either increasing, constant or decreasing. For each investor there are two resulting portfolios, one when the investor considers mean, variance and skewness, and one when the investor only consider mean and variance. The portfolio moments and certainty equivalent are annualized. Euclidean distances are computed on the difference in the weights in assets between the MVS-optimized portfolio, the MV-optimized portfolio and the equally weighted portfolio. Difference in certainty equivalent (CE) is calculated as the CE of the MVS-portfolio less the CE of the MV-portfolio for each investor.

Appendix 3

German large cap portfolios without a riskless asset

Arrow-Pratt Measures			Risk Type		Kane Measures			MVS-Optimized Portfolios				MV-Optimized Portfolios				Euclidean Distances			
RA	dA/dW	dR/dW	ARA	RRA	Skewness Preference	Skewness Ratio	dS/dW	Expected Return	Standard Deviation	Skewness	Certainty Equivalent	Expected Return	Standard Deviation	Skewness	Certainty Equivalent	MVS-MV	MVS-EW	MV-EW	Difference in CE
16	-80	-64	DARA	DRRA	336	21.00	-2368.0	0.1794	0.1276	0.0695	0.0812	0.1935	0.1321	-0.0272	0.0732	0.2124	0.4939	0.4254	0.008037
16	-48	-32	DARA	DRRA	304	19.00	-1120.0	0.1790	0.1271	0.0688	0.0775	0.1907	0.1304	-0.0168	0.0705	0.1998	0.4909	0.4186	0.006946
16	-32	-16	DARA	DRRA	288	18.00	-544.0	0.1788	0.1269	0.0685	0.0756	0.1893	0.1296	0.0134	0.0692	0.1942	0.4895	0.4157	0.006411
16	-32	-16	DARA	DRRA	288	18.00	-560.0	0.1788	0.1269	0.0684	0.0755	0.1893	0.1295	0.0136	0.0691	0.1941	0.4895	0.4157	0.006402
16	-20	-4	DARA	DRRA	276	17.25	-136.0	0.1787	0.1267	0.0682	0.0741	0.1883	0.1290	0.0196	0.0681	0.1903	0.4886	0.4137	0.006013
16	-16	0	DARA	CRRA	272	17.00	0.0	0.1786	0.1266	0.0681	0.0737	0.1880	0.1288	0.0209	0.0678	0.1891	0.4883	0.4130	0.005883
16	-12	4	DARA	IRRA	268	16.75	128.0	0.1786	0.1266	0.0680	0.0732	0.1876	0.1286	0.0221	0.0674	0.1879	0.4880	0.4124	0.005750
16	-12	4	DARA	IRRA	268	16.75	130.0	0.1786	0.1266	0.0680	0.0732	0.1876	0.1286	0.0221	0.0674	0.1879	0.4880	0.4124	0.005751
16	-4	12	DARA	IRRA	260	16.25	384.0	0.1785	0.1265	0.0678	0.0722	0.1870	0.1282	0.0241	0.0667	0.1855	0.4875	0.4112	0.005489
16	0	16	CARA	IRRA	256	16.00	512.0	0.1784	0.1264	0.0677	0.0718	0.1867	0.1280	0.0250	0.0664	0.1844	0.4872	0.4106	0.005361
8	-40	-32	DARA	DRRA	104	13.00	-672.0	0.2174	0.1505	-0.0500	0.1349	0.2196	0.1514	-0.0645	0.1340	0.1049	0.5145	0.4944	0.000826
8	-24	-16	DARA	DRRA	88	11.00	-304.0	0.2155	0.1487	-0.0498	0.1323	0.2194	0.1513	-0.0647	0.1316	0.0901	0.5012	0.4940	0.000729
8	-16	-8	DARA	DRRA	80	10.00	-144.0	0.2147	0.1479	-0.0498	0.1310	0.2193	0.1512	-0.0648	0.1303	0.0846	0.4952	0.4939	0.000701
8	-16	-8	DARA	DRRA	80	10.00	-152.0	0.2147	0.1479	-0.0498	0.1310	0.2193	0.1512	-0.0648	0.1303	0.0845	0.4951	0.4939	0.000699
8	-10	-2	DARA	DRRA	74	9.25	-36.0	0.2141	0.1473	-0.0498	0.1301	0.2192	0.1511	-0.0648	0.1294	0.0814	0.4910	0.4938	0.000688
8	-8	0	DARA	CRRA	72	9.00	0.0	0.2139	0.1471	-0.0498	0.1297	0.2192	0.1511	-0.0649	0.1291	0.0805	0.4896	0.4937	0.000685
8	-6	2	DARA	IRRA	70	8.75	32.0	0.2137	0.1469	-0.0499	0.1294	0.2192	0.1511	-0.0649	0.1287	0.0797	0.4883	0.4937	0.000683
8	-6	2	DARA	IRRA	70	8.75	33.0	0.2137	0.1469	-0.0499	0.1294	0.2192	0.1511	-0.0649	0.1287	0.0797	0.4883	0.4937	0.000684
8	-2	6	DARA	IRRA	66	8.25	96.0	0.2133	0.1466	-0.0499	0.1288	0.2191	0.1511	-0.0649	0.1281	0.0783	0.4857	0.4937	0.000682
8	0	8	CARA	IRRA	64	8.00	128.0	0.2131	0.1464	-0.0499	0.1285	0.2191	0.1511	-0.0649	0.1278	0.0778	0.4845	0.4936	0.000683
4	-20	-16	DARA	DRRA	36	9.00	-208.0	0.2234	0.1573	-0.0531	0.1778	0.2232	0.1565	-0.0626	0.1775	0.0878	0.5468	0.5200	0.000249
4	-12	-8	DARA	DRRA	28	7.00	-88.0	0.2230	0.1565	-0.0556	0.1763	0.2228	0.1559	-0.0626	0.1761	0.0688	0.5359	0.5164	0.000157
4	-8	-4	DARA	DRRA	24	6.00	-40.0	0.2228	0.1561	-0.0567	0.1755	0.2227	0.1557	-0.0627	0.1754	0.0594	0.5309	0.5147	0.000119
4	-8	-4	DARA	DRRA	24	6.00	-44.0	0.2228	0.1561	-0.0568	0.1755	0.2227	0.1557	-0.0627	0.1754	0.0592	0.5308	0.5147	0.000118
4	-5	-1	DARA	DRRA	21	5.25	-10.0	0.2227	0.1559	-0.0576	0.1750	0.2226	0.1555	-0.0627	0.1749	0.0524	0.5274	0.5135	0.000093
4	-4	0	DARA	CRRA	20	5.00	0.0	0.2227	0.1558	-0.0578	0.1748	0.2225	0.1554	-0.0627	0.1747	0.0500	0.5262	0.5132	0.000085
4	-3	1	DARA	IRRA	19	4.75	8.0	0.2226	0.1557	-0.0581	0.1746	0.2225	0.1554	-0.0627	0.1745	0.0477	0.5251	0.5128	0.000078
4	-3	1	DARA	IRRA	19	4.75	8.5	0.2226	0.1557	-0.0581	0.1746	0.2225	0.1554	-0.0627	0.1745	0.0477	0.5251	0.5128	0.000078
4	-1	3	DARA	IRRA	17	4.25	24.0	0.2225	0.1555	-0.0586	0.1742	0.2224	0.1552	-0.0628	0.1741	0.0430	0.5229	0.5120	0.000064
4	0	4	CARA	IRRA	16	4.00	32.0	0.2225	0.1554	-0.0589	0.1740	0.2224	0.1552	-0.0628	0.1740	0.0407	0.5218	0.5117	0.000057

The table provides the result from the optimization of portfolio allocation using DAX30 (large-capitalization German stocks), from a sample of monthly observations between October 2011 and November 2016, not including a risk-free asset, as the asset base. Each row corresponds to one specific investor defined by its risk aversion (RA), derivative of absolute risk aversion (dA/dW), derivative of relative risk aversion (dR/dW), skewness preference, skewness ratio, and derivative of skewness preference (dS/dW). For simplicity the type of risk aversion is included, with ARA meaning absolute risk aversion and RRA meaning relative risk aversion. These can be either increasing, constant or decreasing. For each investor there are two resulting portfolios, one when the investor considers mean, variance and skewness, and one when the investor only consider mean and variance. The portfolio moments and certainty equivalent are annualized. Euclidean distances are computed on the difference in the weights in assets between the MVS-optimized portfolio, the MV-optimized portfolio and the equally weighted portfolio. Difference in certainty equivalent (CE) is calculated as the CE of the MVS-portfolio less the CE of the MV-portfolio for each investor.