STOCKHOLM SCHOOL OF ECONOMICS
Department of Economics
5350 Master's thesis in economics
Academic year 2017–2018

INTERNAL JOB ROTATION AT THE GENERAL SECRETARIAT OF THE COUNCIL OF THE EUROPEAN UNION

- A MATCHING THEORY APPROACH -

KATHERINE PEREZ (41056) AND PASCAL PILLATH (41036)

Abstract: Motivated by the low mobility rates at the General Secretariat of the Council of the European Union (GSC), a job rotation exercise was introduced in 2017, in which participating employees had to switch employment position among themselves. The pilot rotation exhibited a low rate of participation and possible signs of strategic manipulation by the participants. We develop a model that captures the features of a job rotation within an institution and that allows us to determine the causes of the above mentioned problems. We propose two alternative procedures that would improve the outcome of the rotation exercise at the GSC.

Keywords: Market design, matching with contracts, priority matching mechanism, stability, job rotation.

JEL: C78, D47, D61, D78

Supervisor:	Albin Erlanson
Date submitted:	May 13, 2018
Date examined:	May 28, 2018
Discussant:	Agron Nuredini
Examiner:	Mark Sanctuary

ACKNOWLEDGMENTS

We would like to thank Albin Erlanson for his valuable criticism and for being so generous with his time. His help really made a difference. We would also like to thank Jesper Hakansson and Kim Freidberg for their support throughout. Without them, writing this thesis would not have been possible.

CONTENTS

LIST OF FIGURES

LIST OF TABLES

ACRONYMS

AAMC	Association of American Medical Colleges
ACOA	Adjusted Cumulative Offer Algorithm
ACOM	Adjusted Cumulative Offer Mechanism
AD	Administrative
COA	Cumulative Offer Algorithm
COM	Cumulative Offer Mechanism
DG	Directorate-General
EU	European Union
GSC	General Secretariat of the Council of the European Union
HR	Human Resources
IR	Individual Rationality
IR w.r.t. IA	Individual Rationality with respect to the Initial Allocation
NIMP	National Intern Matching Program

1

INTRODUCTION

Markets are often narrowly thought of as places where prices are used to determine who gets what. There are, however, other kinds of markets in which prices play no role, but where exchanges are organized differently. Job markets are an excellent example. You cannot buy your dream job - you must be hired. Similarly, a company will usually not hire solely based on who asks for the lowest wage, but will take other factors like ability and personal fit into account. Only if both sides can come to a mutual agreement, a match is possible. A vast literature has evolved around the organization of such matching markets. In this thesis we use the insights from this literature to examine the functioning of such a market, found at the General Secretariat of the Council of the European Union (GSC).

The GSC, located in Brussels, is the body of around 3000 staff members from all over the European Union (EU) that helps to organize and ensure the coherence of the work of the European Council and the Council of the European Union.[1] As described in its documentation (2018a), the GSC is responsible for a variety of tasks, such as organizing and supporting meetings, preparing and archiving the documents needed for these meetings, providing legal advise and managing the budget for both of the institutions it assists.

The GSC encourages its staff to occasionally change employment position internally (2017d, Art.2). These changes of employment position, referred to as *staff mobility*, bring significant benefits for staff at an individual level as well as for the GSC as a whole. For staff members, mobility offers the chance of experiencing a new working environment, which expands skills and leads to new career prospects and opportunities for personal development. For the GSC, mobility improves efficiency, organizational flexibility and cooperation across units, since teams acquire new skills and perspectives when staff members from different backgrounds join them (2017a; 2017g).

In the past, GSC staff exercised mobility by individually applying to vacant positions within the institution. However, finding an adequate position at the right time was difficult, especially since positions only seldom became vacant, for example when a staff member retired, resulting in mobility rates below the GSC's target (2017a). This motivated the GSC to introduce a yearly job rotation system in 2017. The new rotation system seeks to facilitate mobility by making job changes simultaneous and partially mandatory in order to guarantee a larger pool of vacant positions and job applicants (2017h).

When GSC employees reach a number of years working on the same job,[2] they are required to participate in a job rotation exercise. Additionally, employees are encouraged to participate voluntarily before they complete the maximum number of years they may stay at the same job. The idea of the rotation exercise is that the participating staff members swap jobs among

1 Further details on the institutional background of the GSC can be found in Appendix A.1.
2 The exact duration criteria can be found in Appendix A.3.

themselves, so that at the end everyone has a new job within the institution. Employees participating in the rotation are asked to list the jobs they would like the most out of the existing pool of jobs. Those jobs are overseen by managers, who are also asked to list down their preferred candidates to fill them. Then, the Human Resources (HR) department uses this information to match employees to jobs, following a predefined algorithm (2017h), which we will study in detail in this thesis.

A pilot rotation was scheduled to take place between September of 2017 and July of 2018 (2017i). This pilot rotation exhibited some undesirable features, mainly a low rate of voluntary participation (2017f) and possible signs of strategic manipulation by the participants (2018b). These traits may hint that there is room for improvement in terms of how the rotation exercise is carried out. The goal of this thesis is to find out whether this is the case and, if possible, to propose improvements.

We draw on the tools of matching theory to model and analyze the rotation exercise at the GSC. *Matching theory* is a branch of game theory that studies matching markets, which are markets where, instead of prices determining who gets what, some form of direct exchanges take place (Roth and Sotomayor, 1990). The way in which these direct exchanges are organized significantly influences how well such markets function. A prominent example from the literature that illustrates the importance of matching theory, is the entry-level labor market for American doctors.

In America, a young doctor's first job is an internship at a hospital. As Alvin Roth (1984) describes it, the labor market for interns has always been very competitive on the side of hospitals, since interns are a source of cheap labor and are thus on high demand. In the past, this high level of competition among hospitals to get the best interns induced a number of detrimental practices during recruitment. For example, internship offers from hospitals were being made years before graduation or students were being forced to accept internship offers almost immediately. These practices were harmful both for hospitals and students. Hospitals hiring too soon could not see students' final grades before deciding who to offer positions to, and students were not given enough time to think their decision through. In several instances, the Association of American Medical Colleges (AAMC) introduced new rules aimed at preventing these practices, but none worked permanently.

In an effort to fix the problems, the AAMC launched a voluntary centralized matching procedure in 1950. In this procedure, students would apply to hospitals and be interviewed as usual, and then both students and hospitals would submit rank orders of their most preferred hospitals and students, respectively, to a centralized clearinghouse. This information would then be used to arrange matches between students and hospitals, following a predefined algorithm.

However, this procedure soon sparked opposition among participants. Participants quickly found out that it was possible that a student and a hospital were not matched to each other by the procedure, even though both preferred to be. It only took a student a couple of phone calls to figure out whether one of the hospitals he preferred more than his match, preferred him as well (Roth, 2015). Such a student could then ignore his match and arrange for a better internship outside of the centralized matching procedure, destabilizing the matching arrangements proposed by the clearinghouse. Additionally, students also found out that if they submitted a rank order of hospitals that corresponded to their true preferences, they might receive a less preferable match than if they had submitted a different rank order (Roth and Sotomayor, 1990).

A new algorithm was introduced in 1951: the National Intern Matching Program (NIMP).[3] Participation in this new program increased significantly, with around 95% of graduating doctors each year finding an internship position through it (Roth, 1984). The success of the NIMP could be attributed to the fact that the new algorithm was stable - it was impossible for a student and a hospital to arrange a preferable match outside of the system - and it was strategy-proof - it prevented students from improving their match by submitting rankings different from their true preferences (Roth, 1991).

Markets change constantly, which is why the initial success of the NIMP was not long-lasting. Around the 1970's, participation in the NIMP started to drop again. By that time, a significant number of women had begun studying medicine as well. As a result, the number of graduating doctor couples – medical students married to other medical students – increased drastically. The appearance of couples changed the labor market for interns in a fundamental way. Couples were now making decisions about where to do their internships together. They would, for example, look for internships in the same city so they could live together. By consequence, the satisfaction of one couple member with their assigned internship depended on the internship assigned to his or her partner. The NIMP algorithm was not equipped to deal with such group decision-making and so instabilities arose again (Roth and Sotomayor, 1990). This meant that couples could arrange a better match outside of the centralized procedure, which they did, making participation rates drop once more.

The instabilities introduced by couples in the market were not easy to resolve. The fact that the preferences of some students now depended on what other students got, seriously complicated the type of calculations that the NIMP algorithm was required to do. It took several years of careful examination and trial-and-error attempts to develop the algorithm in such a way that it could deal with such complexities (Roth and Peranson, 1997, 1999).

All this serves to show that what specific procedure is used to organize matches in markets such as the one for medical interns, is crucial. The internal job rotation market at the GSC shares many of the features of the market for medical interns. Just like the young doctors and the hospitals in America, both sides of the internal job rotation market at the GSC - employees and managers - submit preferences to a centralized clearinghouse. The clearinghouse uses a predefined algorithm that takes the preferences of both groups into account to make matches. Consequently, the outcome of the rotation can also be significantly improved or harmed, depending on what algorithm is used. Matching theory provides the framework and tools that allow for a structured study of markets like the one in the GSC, in the hope of constructing a suitable algorithm.

In this thesis, we will use matching theory to answer the question: *Do the observed undesirable features of the rotation exercise at the GSC stem from the currently used algorithm? If yes, can we find an implementable algorithm that leads to improved outcomes?*

Our contribution in this thesis is fourfold. First, we construct a model that captures the features of a job rotation system within an institution. In the matching literature, most labor market models are constructed to represent entry-level labor markets, which display some fundamentally different features. Second, we analyze the rotation exercise at the GSC, which has not been formally analyzed before. Third, we propose two alternative algorithms which address the problems of the currently used algorithm. Fourth, we provide Python scripts that automize the algorithm currently used at the GSC and our proposals.

3 The NIMP eventually changed name and is today known as the National Resident Matching Program (NRMP).

This thesis is structured as follows. In Chapter 2 we introduce some theoretical concepts and background on the relevant literature. In Chapter 3 we present our model of the internal job market at the GSC, which we call the *'worker-manager problem'*. In Chapters 4 and 5 we describe some properties of the worker-manager problem. In Chapter 6 we introduce the concept of a *'mechanism'*, which, in Chapter 7, we use to model and analyze the rotation procedure currently used at the GSC. In Chapter 8 we discuss the shortcomings of the currently used rotation procedure and present some improvement proposals. Chapter 9 concludes.

2

MATCHING WITH CONTRACTS

As the American market for medical interns exemplifies, designing matching markets based on the insights from matching theory has proven successful in the real world. Given a matching problem, like in our case the rotation exercise at the GSC, the first step is to construct a model that captures the situation's relevant features. Such a model is the basic tool that allows for further analysis of the matching problem and that guides any redesign endeavours. The construction of a matching model consists of two parts: modelling agents' decisions and modelling the market in question. In the following sections we will give some background on each.

2.1 MODELLING DECISIONS

Like the students in the market for medical interns who had to choose which internship position to take, agents in matching models face the problem of choosing between a set of mutually exclusive alternatives. There are two approaches to modelling how agents behave when faced with such choice problems: the *preference-based approach* and the *choice-based approach* (Mas-Collel et al., 1995, Ch.1).

2.1.1 *Preference relations*

In the **preference-based approach**, an agent's *tastes* are regarded as his fundamental intrinsic attribute. These tastes can be represented by a **preference relation** over a set of alternatives X, typically denoted by $\succsim$. A preference relation is a binary relation, which means that it allows the direct comparison of pairs of alternatives $a, b \in X$, and is composed of two underlying relations: the *strict-preference relation* and the *indifference relation*. For a pair of alternatives, the **strict-preference relation** $\succ$ dictates that one alternative is liked more than the other, while the **indifference relation** $\sim$ allows for both alternatives to be equally liked. For the most part in economic theory, preference relations are assumed to be **rational**. For a preference relation to be rational, it must be:

1. **complete:** for any $a, b \in X$, either $a \succsim b$ or $b \precsim a$; and
2. **transitive:** for any $a, b, c \in X$, $a \succsim b \ \wedge \ b \succsim c \ \Rightarrow \ a \succsim c$.

2.1.2 *Choice rules*

In the **choice-based approach**, the fundamental object of study is an agent's *choice behavior*, that is, his externally observed decisions, instead of his intrinsic tastes. Formally, an agent's choice behavior, given a set of alternatives X, is characterized by a **choice structure** $(\mathscr{B},\ C(\cdot))$, which is composed of:

1. *a **set of possible choice sets*** $\mathscr{B}$, which is a family of nonempty subsets of X such that each element $B \in \mathscr{B}$ is as subset $B \subseteq X$ and represents a choice situation that an agent might face; and
2. *a **choice rule*** $C(\cdot)$, defined on $\mathscr{B}$, that specifies which alternatives are chosen by an agent for each given subset $B \in \mathscr{B}$. We call the set of alternatives picked by the choice rule the ***'chosen set'***.

Example 2.1. Given a choice set $X = \{a, b, c\}$, a possible subset family is $\mathscr{B} = \{\{a, b\}, \{a, b, c\}\}$ and a choice rule defined over $\mathscr{B}$ would be:

$$C(\{a, b\}) = \{a\} \text{ and } C(\{a, b, c\}) = \{a, b\}.$$

The chosen sets are $\{a\}$ and $\{a, b\}$, respectively.

2.1.3 *The relationship between preference relations and choice rules*

The preference-based approach with the rationality assumption imposes a clear structure on choice behavior that is simple to work with, since it allows to both, write down preferences as ordered lists and to distinctly predict how an agent will behave when faced with different choice problems. However, it is occasionally inadequate to appropriately model the decisions of agents in the real world. Often, just like the couples in the medical interns market, agents find themselves in situations that ask for highly complex decision-making. In such cases, the choice-based approach gives more freedom to accommodate choice behavior that defies being written down as a mere preference list.

Nevertheless, this does not mean that these two approaches are unconnected. In fact, it is always possible to construct a choice structure from a rational preference relation (Mas-Collel et al., 1995, Ch.1). However, given that a choice structure allows for a wider range of choice behavior than does a preference relation, it is only sometimes possible to derive preference relations from choice structures. When this is possible, the derived preference relations are called *'revealed preferences'*.

Example 2.2. *From preference relation to choice structure:* Consider the set of alternatives $X = \{a, b, c\}$ from Example 2.1. If an agent has the rational preference relation $a \succ b \succ c$, we can construct a choice structure where $\mathscr{B} = \{\{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$ and the choice rules would be:

$$C(\{a, b\}) = \{a\}$$
$$C(\{b, c\}) = \{b\}$$

$$C(\{a, c\}) = \{a\}$$

$$C(\{a, b, c\}) = \{a\}$$

From choice structure to preference relation: Consider the choice structure in Example 2.1. From the choice behavior defined by $C(\{a, b\}) = \{a\}$ we can derive the revealed preference relation $a \succ b$. However, we cannot say anything about the preference relation between, for example, a and c. We therefore do not know what the agent would pick, if she is given the choice between a and c.

The model presented in Chapter 3 makes use of both, the preference-based and the choice-based approach to capture the choice behavior of the involved agents.

While modelling how agents make choices is central to matching theory, exactly what choice situations agents will face and what incentives will motivate their choices depend on the market they are in. Modelling such markets is the other main aspect of formulating a matching model.

2.2 MODELLING MARKETS

The theory of matching markets was born in 1962 with the publication of a paper titled *"College Admissions and the Stability of Marriage"* by David Gale and Lloyd Shapley (1962). This paper introduced some central concepts and results that spurred the growth of a large body of literature on matching markets. The field remained mostly theoretical until 1984, when the American market for medical interns was studied for the first time by Alvin Roth, who discovered that the theoretical results of Gale and Shapley surprisingly coincided with the trial-and-error development of the NIMP algorithm (Roth, 1984; Roth and Sotomayor, 1990). For example, Gale and Shapley (1962) had formally defined the concept of 'instability', which exactly captured the idea that students and hospitals could arrange better matches outside of the initial centralized matching procedure we mentioned before. This consistency with the real world added considerable weight to the theory of matching markets and was key to attracting more economists, game theorists in particular, to take an active role in the *design* of matching markets in the real world (Roth, 2002).

Mechanism design,[1] as this discipline has come to be known, reverses the economist's usual approach. In typical economics, economic institutions, for example a market with some particular structure, are taken as given and the objective is to try to predict what economic or social outcomes such institutions will generate. In contrast, the first step in mechanism design is to determine what kind of outcomes are desirable and then try to construct an institution - a mechanism - that can produce such outcomes (Maskin, 2015).

This is not an easy feat. Matching markets in the real world can be very complex in terms of their structure and the kind of decisions that agents have to make, which poses a considerable challenge to mechanism designers. Again, the market for medical interns is an excellent example. Recall that around the mid-1970's, the growing number of couples introduced a higher level of complexity to the matching problem in this labor market (Klaus and Klijn, 2005). Now, on top of having to find matches that were mutually satisfactory for doctor-hospital pairs, a matching mechanism had to also consider group decision-making on the side

1 Note that mechanism design not only specializes in the design of matching markets, but is used for a large variety of other markets including auctions and public goods provision.

of doctors. As mentioned before, many couples wanted to find jobs located in the same city so that they could live together. Because of this, the satisfaction of one doctor with her assigned job depended on what job her partner was assigned. These sorts of complementarities were cumbersome to capture with the existing matching theory language, which relied on the preference-based approach to model agents' decisions. For the members of a couple in the American interns market it was, for example, impossible to submit their true preferences, since separate rankings cannot express the wish of being assigned to the same city.

Another difficulty of the old matching language was that it spoke about direct matches between agents from each side of the market. This also limited the degree of complexity that could be captured by a model. For instance, a market very similar to the one in America exists in the United Kingdom for medical internships. However, in the UK, hospitals distinguish between clinical and research internship positions (Roth, 1991). A hospital might then consider a candidate for a clinical position, but deem him unacceptable for a research position. If we speak of direct matches between hospitals and students, it is impossible to tell which position a student might fill if he is matched to a hospital.

In 2005, John W. Hatfield and Paul Milgrom introduced an alternative way of speaking about matching problems in their paper titled *"Matching with Contracts"*. They presented a novel way of modelling the doctor-hospital problem that generalized some of the existing models and was able to capture much more complex decision situations while remaining relatively straightforward. In the 'matching with contracts' language, matching between agents from each side of the market is no longer done directly, but through *contracts*. In the context of the medical interns market, previous models spoke of doctors having preferences over hospitals and hospitals having preferences over doctors (Klaus and Klijn, 2005; Roth, 1984, 1991). Hatfield and Milgrom instead proposed that both doctors and hospitals have preferences over contracts, where each contract, much like a real life work contract, names a doctor and a hospital and may specify other conditions of employment, such a specific job position or the number of working hours (2005). Agents in a matching with contracts model are 'assigned' a contract, rather than 'matched' to each other, although this represents essentially the same thing.

A further advantage of the matching with contracts language is that, since it mainly uses the choice-based approach to model decisions, it can more easily capture situations where agents make choices over groups of alternatives, like when doctor couples choose between different pairs of jobs or when hospitals wish to hire several doctors to fill a number of different vacancies.

This signified a success for economists studying matching markets in the real world. The years following the publication of the Hatfield & Milgrom paper saw an increase in the practical study of a wide variety of matching markets. The framework has been used to study the Japanese medical-residency matching market (Hatfield et al., 2017; Kamada and Kojima, 2012, 2015, 2017), the matching of cadets to branches of service in the American military (Sönmez, 2013; Sönmez and Switzer, 2013), the allocation of airline seat upgrades (Kominers and Sönmez, 2016), the assignment of legal traineeships in Germany (Dimakopolous and Heller, 2014), the placement of students into graduate degrees in psychology in Israel (Hassidim et al., 2017), and the matriculation of students into the Indian Institutes of Technology (Aygün and Turhan, 2017).

In this thesis we also use the matching with contracts language to model the internal job market at the GSC. The model is presented in the next chapter.

3

THE MODEL

The model presented in this chapter captures the job rotation market at the GSC. This model is meant to portray a labor market within a public institution and thus differs in some crucial aspects from other models in the matching literature that target classical labor markets were workers seek employment from firms (Klaus and Klijn, 2005; Roth, 1984, 1991; Sönmez and Switzer, 2013). For example, the workers in our model are already employed within the institution, while workers in other models are usually not assumed to be. Additionally, positions in our model are heterogeneous, while positions at firms in other models are often homogeneous.

To ease understanding we develop an extensive example throughout this chapter. We will refer back to this example in later parts of this thesis as well.

3.1 SETS OF PARTICIPANTS

The base of the model consists of workers, managers and jobs. There are r **workers**, denoted $W = \{w_1, w_2, ..., w_r\}$ and s **managers**, denoted $M = \{m_1, m_2, ..., m_s\}$. Each manager m oversees a certain number q_m of **jobs**, denoted by $J_m = \{j_1^m,\ ...,\ j_{q_m}^m\}$. We call q_m manager m's **capacity**. Since each worker has a job, there are at least as many jobs as workers. Formally, $\sum_{m \in M} q_m \geq r$. We denote the set of all jobs by $J = \bigcup_{m \in M} J_m$.

3.2 CONTRACTS

A **contract** $x = (w, j) \in W \times J$ specifies a worker w and a job j. For a given contract $x = (w, j)$, let:

- x_W be the worker w named in x;
- x_J be the job j named in x; and
- x_M be the manager who oversees job j.

Let $X \equiv W \times J$ be the **set of all contracts**. This set contains all possible one-to-one combinations of workers and jobs. For a given set of contracts $X' \subseteq X$, let:

- $X'(w) \equiv \{x \in X' : x_W = w\}$ be the set of all contracts that name worker w;
- $X'(j) \equiv \{x \in X' : x_J = j\}$ be the set of all contracts that name job j; and
- $X'(m) \equiv \bigcup_{j \in J_m} X'(j)$ be the set of all contracts that involve manager m.

Furthermore, let:

- $X'_W \equiv \{x_W \in W : \ x \in X'\}$ be the set of all workers named in some contract in X';
- $X'_J \equiv \{x_J \in J : \ x \in X'\}$ be the set of all jobs named in some contract in X'; and
- $X'_M \equiv \{x_M \in M : \ x \in X'\}$ be the set of all managers who oversee the jobs named in the contracts in X'.

Example 3.1. Let there be four workers denoted by $W = \{w_1, w_2, w_3, w_4\}$ and two managers $M = \{m_1, m_2\}$. Let manager m_1 have two jobs and manager m_2 have three jobs, so that there are five jobs in total: $J = \{j_1^{m_1},\ j_2^{m_1}\} \cup \{j_1^{m_2},\ j_2^{m_2},\ j_3^{m_2}\}$. The set of all contracts X can be visualised in a 4×5 matrix where each element represents one contract:

X	$j_1^{m_1}$	$j_2^{m_1}$	$j_1^{m_2}$	$j_2^{m_2}$	$j_3^{m_2}$
w_1	$(w_1, j_1^{m_1})$	$(w_1, j_2^{m_1})$	$(w_1, j_1^{m_2})$	$(w_1, j_2^{m_2})$	$(w_1, j_3^{m_2})$
w_2	$(w_2, j_1^{m_1})$	$(w_2, j_2^{m_1})$	$(w_2, j_1^{m_2})$	$(w_2, j_2^{m_2})$	$(w_2, j_3^{m_2})$
w_3	$(w_3, j_1^{m_1})$	$(w_3, j_2^{m_1})$	$(w_3, j_1^{m_2})$	$(w_3, j_2^{m_2})$	$(w_3, j_3^{m_2})$
w_4	$(w_4, j_1^{m_1})$	$(w_4, j_2^{m_1})$	$(w_4, j_1^{m_2})$	$(w_4, j_2^{m_2})$	$(w_4, j_3^{m_2})$

The contract $x = (w_1, j_1^{m_2})$ includes worker w_1, manager m_2 and manager m_2's first job $j_1^{m_2}$. Here, $x_W = w_1$, $x_J = j_1^{m_2}$ and $x_M = m_2$. Now consider another contract $x' = (w_2, j_3^{m_2})$. This contract includes worker w_2, manager m_2 as well, but this time m_2's third job $j_3^{m_2}$. Here, $x'_W = w_2$, $x'_J = j_3^{m_2}$ and $x'_M = m_2$. For worker w_1, the set of all contracts that name her is:

$$X(w_1) = \{(w_1, j_1^{m_1}), (w_1, j_2^{m_1}), (w_1, j_1^{m_2}), (w_1, j_2^{m_2}), (w_1, j_3^{m_2})\}.$$

The set of all workers named in the set of all contracts X is simply:

$$X_W = \{w_1, w_2, w_3, w_4\}.$$

3.3 ALLOCATIONS

An **allocation** $A \subseteq X$ is a set of contracts such that:[1]

- each worker appears in at most one contract $|A(w)| \leq 1,\ \forall w \in W$;
- each job appears in at most one contract $|A(j)| \leq 1,\ \forall j \in J$; and
- no manager appears in more contracts than her capacity $|A(m)| \leq q_m, \forall m \in M$.

Let $\mathscr{A}$ denote the set of all possible allocations. The **initial allocation** $A^0 \subseteq X$ is the set of contracts already in place at the beginning of a rotation exercise. We call worker w's assignment under the initial allocation his **initial contract**, denoted by $A^0(w) = x^0$.

1 With $|A|$ we denote the cardinality of A.

3.4 PREFERENCES AND CHOICE RULES

Just like in the GSC, most workplaces host a number of different job roles. Job roles might differ in terms of the responsibilities they entail, the required skills needed to perform them, the perks attached to them and many other dimensions. Given these differences, it is natural that workers prefer some jobs over others. At the same time, workers are different, too. Workers can differ in terms of their field of expertise, their experience and so on, making some workers more suitable for certain jobs than others. While managers are happy to see the jobs they oversee filled, they have to take these differences into account when deciding who to hire.

In the following sections we will take a closer look at the preferences of the workers and the preferences of the managers and explain the assumptions we make about them.

3.4.1 *Workers' preferences*

In our context, for the workers there is no practical difference between having a certain job and having the contract connected to that job. For this reason we can say that workers have *preferences* over contracts. We assume that worker's preferences are *rational* and *strict*. They can thus be described by a strict total ordering over all contracts and can be completely represented by a preference list, as explained in Chapter 2. We denote this as the **preference relation $\boldsymbol{P_w}$** of worker $w \in W$ over the set $X(w) \cup \{\varnothing\}$. We write P_w and R_w when comparing pairs of contracts. For example, looking at three contracts x, y and z, if we write $x \ P_w \ y \ P_w \ z$, we mean that worker w prefers contract x over contract y, and both contract x and contract y over contract z. If we write $x \ R_w \ y$, we mean that worker w weakly prefers contract x over contract y.[2]

Furthermore, we assume that each worker w ranks a 'null contract' $\varnothing$, which represents not having a job at the public institution, in our case the GSC. We say that a contract x is **acceptable** to worker w if $x \ P_w \ \varnothing$. A contract y is **unacceptable** to worker w if $\varnothing \ P_w \ y$. We assume that the initial contract is always acceptable, that is $A^0(w) \ P_w \ \varnothing$. For ease of notation, we use the convention that each worker prefers the null contract over every contract that does not name him, formally: $\varnothing \ P_w \ x$ if $x \in X \setminus X(w)$.

We assume that a worker only cares about the contract that she holds, that is, a worker's assigned contract under a particular allocation determines her preferences over allocations. This means that a worker w prefers an allocation A to another allocation A' if and only if she prefers $A(w)$ to $A'(w)$. To keep our notation simple, we use P_w to denote worker w's preferences over allocations as well as her preferences over contracts X.

We denote a profile of all workers' preferences by $P_W \equiv (P_w)_{w \in W}$. Similarly, we denote a profile of all workers' preferences except those of worker w by $P_{-w} \equiv (P_{w'})_{w' \in W \setminus \{w\}}$. Moreover, we denote the set of all possible preferences of worker w as $\mathcal{P}_w$ and the set of all possible profiles of all workers' preferences as $\mathcal{P}_W$. Workers have *unit demand*, which means that they choose at most one contract - their most preferred one - out of a set of available contracts.

Recall from Chapter 2 that it is possible to construct a choice structure from a rational preference relation. Using our assumption that workers have a strict rational preference

2 The notation P_w is equivalent to $\succ$ as defined in Chapter 2. We write R_w to express the weak preference relation $\succeq$.

relation P_w over contracts, we can write a choice structure $(\mathscr{B}, C_w(\cdot))$ for the worker's choice behavior that is generated by P_w, as follows:

- Let $\mathscr{B}$ be the set of choice sets over contracts that a worker w might face, composed of all possible subsets of X.
- Let $C_w(\cdot)$ be the **choice rule of a worker** $w \in W$ such that from a set of contracts $X' \subseteq X$, the **chosen set of a worker** is:

$$C_w(X') \equiv \max_{P_w} X'$$

The choice rule simply states that a worker will choose his most preferred contract out of the subset X' according to his preference relation P_w. The chosen set of a worker will always be a singleton.

Example 3.2. Let the initial allocation be:

$$A^0 = \left\{ \left(w_1, j_1^{m_1}\right)^0, \ \left(w_2, j_2^{m_1}\right)^0, \ \left(w_3, j_1^{m_2}\right)^0, \ \left(w_4, j_2^{m_2}\right)^0 \right\}.$$

The preferences of workers w_1, w_2, w_3 and w_4 over the contracts in X from Example 3.1 can be listed as follows:

P_{w_1}	P_{w_2}	P_{w_3}	P_{w_4}
$\left(w_1, j_2^{m_2}\right)$	$\left(w_2, j_2^{m_2}\right)$	$\left(w_3, j_1^{m_1}\right)$	$\left(w_4, j_1^{m_2}\right)$
$\left(w_1, j_3^{m_2}\right)$	$\left(w_2, j_1^{m_1}\right)$	$\left(w_3, j_2^{m_2}\right)$	$\left(w_4, j_3^{m_2}\right)$
$\left(w_1, j_1^{m_1}\right)^0$	$\left(w_2, j_2^{m_1}\right)^0$	$\left(w_3, j_3^{m_2}\right)$	$\left(w_4, j_1^{m_1}\right)$
$\left(w_1, j_1^{m_2}\right)$	$\left(w_2, j_3^{m_2}\right)$	$\left(w_3, j_2^{m_1}\right)$	$\left(w_4, j_2^{m_1}\right)$
$\varnothing$	$\left(w_2, j_1^{m_2}\right)$	$\left(w_3, j_1^{m_2}\right)^0$	$\left(w_4, j_2^{m_2}\right)^0$
$\left(w_1, j_2^{m_1}\right)$	$\varnothing$	$\varnothing$	$\varnothing$

Given the set of all contracts X, the chosen set for each worker is then:

$$C_{w_1}(X) = \left\{\left(w_1, j_2^{m_2}\right)\right\} \quad C_{w_2}(X) = \left\{\left(w_2, j_2^{m_2}\right)\right\}$$
$$C_{w_3}(X) = \left\{\left(w_3, j_1^{m_1}\right)\right\} \quad C_{w_4}(X) = \left\{\left(w_4, j_1^{m_2}\right)\right\}$$

3.4.2 *Managers' preferences*

Just like with the workers, we assume that managers' preferences are rational. In contrast to the workers, however, managers' preferences differ in one crucial aspect: a manager might oversee *multiple* jobs simultaneously, whereas a worker can only fill one job at a time, so that managers do not necessarily have unit demand. We say that manager $m \in M$ has the **preference relation** $\boldsymbol{P_m}$ over $\{Y \mid Y \subseteq X(m) \cup \{\varnothing\}\}$. Similar to the workers, we assume that managers also rank the empty set. For the managers, this means leaving all of their jobs unfilled. To keep our notation simple, we use P_m to denote manager m's preferences over sets of contracts as well as his preferences over allocations. We denote the set of all managers' preferences by $P_M \equiv (P_m)_{m \in M}$.

Just like the workers' preferences, manager preferences are likely to display some natural regularities. However, modelling the preferences of managers can be very difficult. Imagine that we were to ask a manager who oversees several jobs to list down his preferences over all possible groups of workers to fill those jobs. He might be able to rank a few possible groups, but most likely not all. Consider a manager who oversees five jobs and has to pick a group of five workers to fill them. Assume that there are ten candidates. Each of these workers only has to rank five contracts. The manager, however, faces 30,240 possible combinations of workers to fill the five jobs. It is implausible to assume that the manager would be able to rank all of these possibilities. We therefore use the choice-based approach to model the choice behavior of the managers.[3]

We base the managers' choice structure on two assumptions. The first one is that *some jobs are more important than others*. In the GSC, each job is bound to a certain thematic area (2017e). Depending on current events, some thematic areas can be assumed to be more important than others. Given this, it is plausible to say that a job bound to an important thematic area has a higher priority to its manager than another job bound to a less important thematic area. The second and more intuitive assumption is that *some workers are more suitable for some jobs than others.*

Formally, these two assumptions can be described as follows:

1. *each manager has a **precedence order** $\triangleright_m$ over the jobs she oversees*, i.e. over J_m. Given two jobs $j, j' \in J_m$, $j \triangleright_m j'$ states that whenever possible, manager m fills job j before filling job j'. We use the convention that, for a set of jobs overseen by the same manager $J_m = \{j_1^m, ..., j_{q_m}^m\} \Rightarrow j_n^m \triangleright j_{n+1}^m$, unless otherwise specified.
2. *each job has a **priority order** $\mathbf{\Pi_j}$ over the contracts that name it*, i.e. over $X(j) \cup \{\varnothing\}$. We use Π_j to compare pairs of contracts. We say that a contract x is **acceptable** for job j if $x \; \Pi_j \; \varnothing$. We say that a contract y is **unacceptable** for job j if $\varnothing \; \Pi_j \; y$. We assume that the initial contract is always acceptable, that is $A^0(j) \; \Pi_j \; \varnothing$. We denote the profile of all jobs' priorities by $\Pi_J \equiv (\Pi_j)_{j \in J}$.

Before we can turn to the choice structure of the managers, we need to define the **choice structure for each job**.

- Let $\mathscr{B}$ be the set of choice sets that a job j might face, composed of all possible subsets of X. Note that this is the same set of choice sets as for the workers.
- Let $C_j(\cdot)$ be the **priority rule of a job** $j \in J$ from a set of contracts $X' \subseteq X$. The priority rule simply chooses the contract with the highest priority according to the priority order Π_j:
$$C_j(X') \equiv \max_{\Pi_j} \; X'$$

Using the choice structure of the jobs and the precedence order we now model the choice behavior of managers so that a manager first chooses the best worker for her most important job, then continues to choose the best available worker for her next most important job and so on. Formally, we define the **choice structure of the managers** as follows:

- Let $\mathscr{B}$ be the set of choice sets over contracts that a manager m might face, composed of all possible subsets of X. This is the same set of choice sets as for the workers and the jobs.

3 Our way of modelling the managers' preferences is based on the model of branch preferences in Kominers and Sönmez (2016).

- Let $C_m(\cdot)$ be the **choice rule of a manager** $m \in M$. Given a set $X' \subseteq X$ the rule defines the **chosen set of a manager** $C_m(X')$ in the following way:

STEP 1: Job j_1^m is assigned its most preferred contract $x^1 = C_{j_1^m}(X')$ according to $\Pi_{j_1^m}$ among all available contracts in X'.

STEP 2: Job j_2^m is assigned its most preferred contract $x^2 = C_{j_2^m}(X')$ according to $\Pi_{j_2^m}$ among all available contracts in $X' \setminus X'(x_W^1)$.

STEP k: Job j_k^m is assigned its most preferred contract $x^k = C_{j_k^m}(X')$ according to $\Pi_{j_k^m}$ among all available contracts in $X' \setminus \bigcup_{l=1}^{k-1} X'(x_W^l)$.

There are q_m steps in all. The **chosen set of a manager** is $C_m(X') = \bigcup_{l=1}^{q_m} C_{j_l^m}(X')$.

This choice structure specifies how a manager behaves in any situation he might face. Therefore we can derive a revealed preference relation that coincides with the preference relation P_m of the managers. For convenience, we denote this revealed preference relation also as P_m.

Example 3.3. Assume the following priority orders for the managers m_1 and m_2 over the contracts from Example 3.1:

$\Pi_{j_1^{m_1}}$	$\Pi_{j_2^{m_1}}$	$\Pi_{j_1^{m_2}}$	$\Pi_{j_2^{m_2}}$	$\Pi_{j_3^{m_2}}$
$(w_2, j_1^{m_1})$	$(w_3, j_2^{m_1})$	$(w_2, j_1^{m_2})$	$(w_1, j_2^{m_2})$	$(w_2, j_3^{m_2})$
$(w_3, j_1^{m_1})$	$(w_1, j_2^{m_1})$	$(w_1, j_1^{m_2})$	$(w_2, j_2^{m_2})$	$(w_1, j_3^{m_2})$
$(w_4, j_1^{m_1})$	$(w_4, j_2^{m_1})$	$(w_4, j_1^{m_2})$	$(w_3, j_2^{m_2})$	$(w_4, j_3^{m_2})$
$(w_1, j_1^{m_1})^0$	$(w_2, j_2^{m_1})^0$	$(w_3, j_1^{m_2})^0$	$(w_4, j_2^{m_2})^0$	$(w_3, j_3^{m_2})$
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$

Given that the managers can choose from the set of all possible contracts X from Example 3.1, we can use their choice rule to determine their chosen sets.

Consider the manager m_1. In the first step, the most preferred contract for her first job $j_1^{m_1}$ is determined: $C_{j_1^{m_1}}(X) = \{(w_2, j_1^{m_1})\}$. In the second step, the most preferred contract for her second job is determined: $C_{j_2^{m_1}}(X) = \{(w_3, j_2^{m_1})\}$. Since manager m_1 only has two jobs to oversee, the choice rule ends here. The chosen set is therefore

$$C_{m_1}(X) = C_{j_1^{m_1}}(X) \cup C_{j_2^{m_1}}(X) = \{(w_2, j_1^{m_1}), (w_3, j_2^{m_1})\}$$

Now consider manager m_2. In the first step, the most preferred contract for her first job $j_1^{m_2}$ is determined: $C_{j_1^{m_2}}(X) = \{(w_2, j_1^{m_2})\}$. In the second step, the most preferred contract for her second job $j_2^{m_2}$ is determined: $C_{j_2^{m_2}}(X) = \{(w_1, j_2^{m_2})\}$. In the third step however, we notice that worker w_2 and w_1 were already assigned to a job by manager m_2 so that their contracts for $j_3^{m_2}$ are no longer available. Hence the most preferred available contract is $C_{j_3^{m_2}}(X) = \{(w_4, j_3^{m_2})\}$. It follows that m_2's chosen set is:

$$C_{m_2}(X) = C_{j_1^{m_2}}(X) \cup C_{j_2^{m_2}}(X) \cup C_{j_3^{m_2}}(X) = \left\{ (w_2, j_1^{m_2}), (w_1, j_2^{m_2}), (w_4, j_3^{m_2}) \right\}$$

This concludes the description of the model.

3.4.3 *Parallels between workers' and managers' preferences*

Table 3.1 illustrates the parallels between the preferences of the workers and the managers and how we constructed them. Our initial assumption for the workers was that they have rational preferences. This assumption allowed us to formulate a choice structure, composed of choice sets and a choice rule. Our approach to modelling the managers' preferences went in the opposite direction, where our initial assumption was that managers make decisions according to a precedence order over the jobs they oversee and a priority order over candidates for each job. From this choice structure it is possible to derive the revealed preferences for the managers.

	Workers	Managers
Assumption	Rational preferences (P_w)	Choice structure $(\mathscr{B}, C_m(\cdot))$ based on a precedence order $(\triangleright_m)$ and a priority order (Π_j)
Derivation	Choice structure $(\mathscr{B}, C_w(\cdot))$	Revealed preferences (P_m)

Table 3.1: Parallels between workers' and managers' preferences

3.5 THE WORKER-MANAGER PROBLEM

In this chapter we presented a model for the labor rotation market at the GSC. From now on we refer to this model as the **worker-manager problem**, which consists of:

1. a finite set of workers $W = \{w_1, w_2, ...w_r\}$
2. a finite set of managers $M = \{m_1, m_2, ...m_s\}$
3. a list of capacities $q = (q_m)_{m \in M}$ with $\sum_{m \in M} q_m \geq r$
4. a finite set of jobs $J = \bigcup_{m \in M} J_m = \{j_1^{m_1}, ..., j_{q_{m_1}}^{m_1}, j_1^{m_2}, ..., j_{q_{m_2}}^{m_2}, ..., j_1^{m_s}, ..., j_{q_{m_s}}^{m_s}\}$
5. a list of worker preferences $P_W = (P_w)_{w \in W}$ over $X(w) \cup \{\varnothing\}$,
6. a list of revealed manager preferences $P_M = (P_m)_{m \in M}$ over $X(m) \cup \{\varnothing\}$,
7. a list of job priorities $\Pi_J = (\Pi_j)_{j \in J}$ over $X(j) \cup \{\varnothing\}$, and
8. a list of job precedences $(\triangleright_m)_{m \in M}$ over J_m

In the next chapter we explore a concept central to the matching literature: stability.

4

STABILITY IN THE WORKER-MANAGER-PROBLEM

Stability is one of the most important concepts in matching theory. The idea of stability is that no worker or manager, alone or as a pair, has the incentive and the ability to disrupt an allocation. To ensure that an allocation is stable, two things must hold: first, every worker and manager must be assigned a contract that they find better than having no contract at all and second, there may not exist any workers and managers that would rather have a contract with each other than the contracts they currently have.

The importance of stability has been widely evidenced in practice. As mentioned in Chapter 1, the success of the NIMP algorithm before couples entered the market is attributed to the fact that it produced stable allocations. Similarly, in the UK, the procedures used in medical intern markets that proved the most long-lasting were those which produced stable allocations, while those who did not were usually abolished after a few years due to low participation rates and participant dissatisfaction (Roth, 1991). This is why in this chapter we will formally define stability for the worker-manager problem and show that we can always find a stable allocation.

4.1 STABLE ALLOCATIONS IN THE WORKER-MANAGER PROBLEM

Definition 4.1. *(Kominers and Sönmez, 2016) An allocation A is **stable** if it is:*

*(i) **Individually rational:** We have $C_w(A) = A(w)$ for all $w \in W$ and $C_m(A) = A(m)$ for all $m \in M$; and*

*(ii) **Unblocked:** There do not exist a manager $m \in M$ and blocking set of contracts $B \neq C_m(A)$ such that $B = C_m(A \cup B)$ and $B(w) = C_w(A \cup B)$ for all $w \in B(w)$.*

In words, individual rationality requires that no worker or manager is assigned a contract that he finds unacceptable. Given that A is an allocation, per definition $A(w)$ must be a set of at most one contract for each worker w. If this contract is acceptable to worker w, he will choose it over the option of having no contract, i.e. if $A(w)\ P_w\ \varnothing$ then $C_w(A) = A(w)$. If the worker would prefer $\varnothing$ over $A(w)$, then $C_w(A) = \varnothing$. The same logic applies to the managers. Per definition, $A(m)$ can at most have q_m contracts in total and at most one contract for each specific job of m. Otherwise A would not be an allocation. If each contract in $A(m)$ is acceptable for manager m, we must have that $C_m(A) = A(m)$.

If an allocation A is unblocked, we cannot find a set of contracts B naming one manager m and one or more workers $(w_i)_{i \in B(w) \subseteq W}$ such that all of them would choose the contracts that name them in B rather than their assigned contracts under the allocation A.

Example 4.1. Recall the preferences of the workers $W = \{w_1, w_2, w_3, w_4\}$ in Example 3.3 and the preferences of the managers $M = \{m_1, m_2\}$ in Example 3.4. Using those preferences, let us examine two different allocations.

The allocation $A = \{(w_1, j_2^{m_1}), (w_2, j_2^{m_2}), (w_3, j_3^{m_2}), (w_4, j_1^{m_1})\}$ violates individual rationality, since worker w_1 is assigned a contract that she finds unacceptable. Therefore, she would rather have no contract than the one she is assigned to under the allocation A: $C_{w_1}(A) = \varnothing \neq A(w_1)$. Thus, allocation A is unstable.

Another allocation $A' = \{(w_1, j_2^{m_2}), (w_2, j_3^{m_2}), (w_3, j_1^{m_1}), (w_4, j_2^{m_1})\}$ satisfies individual rationality for all participants. However, it is blocked. To see why, consider worker w_2 and manager m_1. Both would prefer to be matched to each other rather than being with their assignment under A':

$$(w_2, j_1^{m_1})\ P_{w_2}\ (w_2, j_3^{m_2})$$

and

$$\{(w_2, j_1^{m_1}), (w_4, j_2^{m_1})\}\ P_{m_1}\ \{(w_3, j_1^{m_1}), (w_4, j_2^{m_1})\}.$$

Formally, let $B = \{(w_2, j_1^{m_1}), (w_4, j_2^{m_1})\}$. Therefore we have that $(A' \cup B)$ is equal to A' with the additional element $(w_2, j_1^{m_1})$. Hence,

$$(A' \cup B) = \{(w_1, j_2^{m_2}), (w_2, j_3^{m_2}), (w_3, j_1^{m_1}), (w_4, j_2^{m_1}), (w_2, j_1^{m_1})\}$$

One can now check that $C_{m_1}(A' \cup B) = B$ and $C_{w_1}(A' \cup B) = B(w_1)$. Thus, w_2 and m_1 block allocation A'.

In the following sections we will prove that there always exist stable allocations in the worker-manager problem. To do this we first show that some conditions for stability commonly used in the literature of matching with contracts are satisfied in the worker-manager problem. Then we show that these conditions are sufficient to guarantee the existence of stable allocations.

4.2 CONDITIONS FOR STABILITY

The idea behind the conditions for stability seen in the matching with contracts literature is that when both sides of the market regard each other more as substitutes than as complements, then it can be shown that stable allocations exist. We therefore examine whether the participants in the worker-manager problem consider the other side of the market more as substitutes than complements. Regarding the workers, this can easily be answered. Since workers have unit demand over contracts, jobs will always be substitutes for them. If one job is no longer available, then a worker will simply pick another job. For the managers, however, it is not as straight-forward, since they have preferences over groups. It is not clear whether a group of workers employed under a manager are rather substitutes or complements. To illustrate, consider a rowing team and an American football team. In a rowing team, if one oarsman drops out, every remaining oarsman could still be regarded as a useful part of the team. We might say that oarsmen are substitutes. However, if a player who can only catch long passes in a football team drops out, then a player who can only throw such long passes might loose some of his usefulness within the team. We might say that the thrower and the catcher are complements. In this sense, it is not clear whether a group of workers employed by a manager are like a rowing team, a football team or something in between.

Several different notions have been proposed to capture the spectrum of substitutability (Aygün and Sönmez, 2012; Hatfield and Kojima, 2010; Hatfield and Milgrom, 2005; Roth and Sotomayor, 1990). In our case, it is possible to show that, due to our assumptions on their choice structure, managers regard workers as bilateral substitutes, which is a rather weak notion of substitutability. Using this insight together with the *irrelevance of rejected contracts (IRC)*, we can prove that stable allocations always exist in the worker-manager problem. This approach was originally introduced by Aygün and Sönmez (2012) and Hatfield and Kojima (2010) for the student-hospital problem from Hatfield and Milgrom (2005).

Below we prove that IRC and bilateral substitutability are satisfied for the choice rule of the managers in the worker-manager problem. For the sake of completeness we also show that two other notions of substitutability, namely *substitutability* and *unilateral substitutability* and two commonly used conditions, the *Law of Aggregate Demand* and *Pareto Separability*, are not satisfied by the managers' choice rule.

Condition 1: Irrelevance of Rejected Contracts

Definition 4.2. *(Aygün and Sönmez, 2012) A choice rule C_m satisfies* ***irrelevance of rejected contracts (IRC)*** *if for all $Y \subseteq X$ and $x \in X \setminus Y$,*

$$x \notin C_m(Y \cup \{x\}) \implies C_m(Y \cup \{x\}) \;=\; C_m(Y).$$

In words, if a manager does not choose a contract from some set of available contracts, then the chosen set will be the same whether that specific contract is available or not. This means that the removal of rejected contracts shall not affect chosen sets.

Lemma 4.1. *C_m satisfies IRC.*

In order to prove Lemma 4.1, we follow Kominers and Sönmez (2016, Proof of Lemma 1). That is, we prove the following claim, which states that exactly the same contracts are assigned to jobs of m in the computation of $C_m(Y \cup \{x, x'\})$ and $C_m(Y \cup \{x\})$. This Claim directly implies our Lemma 4.1.

Claim 1. *Suppose that $x' \notin C_m(Y \cup \{x, x'\})$, and for each l with $1 \le l \le q_m$, let z^l and y^l be the contracts assigned to j_l^m in the computations of $C_m(Y \cup \{x, x'\})$ and $C_m(Y \cup \{x\})$. We have $z^l = y^l$.*

Proof of Claim 1. Suppose that $x' \notin C_m(Y \cup \{x, x'\})$ and proceed by induction. First, we have

$$z^1 = \max_{\Pi_{j_1^m}} (Y \cup \{x, x'\})$$
$$y^1 = \max_{\Pi_{j_1^m}} (Y \cup \{x\})$$

by definition. As $x' \notin C_m(Y \cup \{x, x'\})$, we know in particular that $z^1 \neq x'$, so we must have

$$z^1 = \max_{\Pi_{j_1^m}} (Y \cup \{x, x'\}) = \max_{\Pi_{j_1^m}} (Y \cup \{x\}) = y^1$$

Thus, we suppose that $z^{l'} = y^{l'}$ for all $l' < l$ (with $l < q_m$). Now, again by definition, we have:[1]

$$y^l = \max_{\Pi_{j_l^m}} (Y \cup \{x\} \setminus Y(\{y_W^1, y_W^2 ..., y_W^{l-1}\})) \tag{4.1}$$

As $x' \notin C_m(Y \cup \{x, x'\})$, we must have $z^l \neq x'$, so that

$$\begin{aligned} z^l &= \max_{\Pi_{j_l^m}} (Y \cup \{x, x'\} \setminus Y(\{z_W^1, z_W^2 ..., z_W^{l-1}\})) \\ &= \max_{\Pi_{j_l^m}} (Y \cup \{x\} \setminus Y(\{z_W^1, z_W^2 ..., z_W^{l-1}\})) \\ &= \max_{\Pi_{j_l^m}} (Y \cup \{x\} \setminus Y(\{y_W^1, y_W^2 ..., y_W^{l-1}\})) \\ &= y^l \end{aligned}$$

where the second equality follows from the fact that $z^l \neq x'$, the third equality follows from the inductive hypothesis, and the last equality follows from equation (4.1). Thus, we have shown that if $z^{l'} = y^{l'}$ for all $l' < l$, then we have $z^l = y^l$; this completes our induction. Hence C_m satisfies IRC. □

Condition 2: Substitutability

Definition 4.3. *(Schlegel, 2015) A choice function C_m satisfies **substitutability** if $X' \subseteq X$ and for all $x, z \in X \setminus X'$:*

$$x \in C_m(X' \cup \{x, z\}) \implies x \in C_m(X' \cup \{x\}).$$

In words, if a manager chooses a contract x from a set of contracts, then she also chooses x from any subset that contains x.

Claim 2. *C_m does not satisfy substitutability.*

Proof of Claim 2. We prove by example.

Consider manager m_2 from Example 3.3. Let $X' = \{(w_4, j_1^{m_2}), (w_2, j_3^{m_2})\}$, $x = (w_1, j_3^{m_2})$ and $z = (w_2, j_1^{m_2})$. From the manager's choice rule it follows that:

$$\begin{aligned} C_{m_2}(X' \cup \{x, z\}) &= \{(w_2, j_1^{m_2}), (w_1, j_3^{m_2})\} \\ C_{m_2}(X' \cup \{x\}) &= \{(w_4, j_1^{m_2}), (w_2, j_3^{m_2})\} \end{aligned}$$

We can see that x is included in $C_{m_2}(X' \cup \{x, z\})$ but not in $C_{m_2}(X' \cup \{x\})$. Hence C_m does not satisfy substitutability. □

1 Similar to the definitions in chapter 3, we write: given a set $W' \in W$, let $X'(W') \equiv \{x \in X' : x_W \in W'\}$.

Condition 3: Unilateral Substitutability

Definition 4.4. *(Schlegel, 2015) A choice function C_m satisfies* ***unilateral substitutability*** *if for each $X' \subseteq X$ and for all $x, z \in X \setminus X'$ with $x_W \notin X'_W$:*

$$x \in C_m(X' \cup \{x, z\}) \Rightarrow x \in C_m(X' \cup \{x\}).$$

In words, whenever a manager m accepts contract x when that is the only contract with worker x_W available, she still accepts x when the choice set shrinks and x is still available.

Claim 3. *C_m does not satisfy unilateral substitutability.*

Claim 3 can be proven with the same example used to prove claim 2. Hence, C_m does not satisfy unilateral substitutability.

Condition 4: Bilateral Substitutability

Definition 4.5. *(Schlegel, 2015) A choice function C_m satisfies* ***bilateral substitutability*** *if for each $X' \subseteq X$ and for all $x, z \in X \setminus X'$ with $x_W, z_W \notin X'_W$:*

$$x \in C_m(X' \cup \{x, z\}) \Rightarrow x \in C_m(X' \cup \{x\}).$$

In words, let x and z be two contracts such that they are the only contracts that name the workers x_W and z_W. Whenever a manager m accepts contract x, she still accepts the contract x when z is no longer available but x still is.

Proposition 4.1. *C_m satisfies bilateral substitutability.*

Proof of Proposition 4.1. Given that $x \in C_m(X' \cup \{x, z\})$, there are two cases:

CASE 1: $z \notin C_m(X' \cup \{x, z\})$: If z was not chosen when it was available, the chosen set will stay the same when z becomes unavailable, because of IRC, which is satisfied by the choice rule of the managers (Lemma 4.1).

CASE 2: $z \in C_m(X' \cup \{x, z\})$ If z is chosen, there are two sub cases to consider:

- *z's job precedence is lower than x's job precedence.* When z becomes unavailable, x will still be chosen because x_J and all jobs with a higher precedence than x_J can choose from the same set and hence will chose the same contracts.
- *z's job precedence is higher than x's job precedence.* The only case when x will not get chosen at $X' \cup \{x\}$ is if a contract with a higher priority than x becomes available at job x_J. However, this cannot happen when z becomes unavailable because the worker z_W only has exactly one contract in $X' \cup \{x, z\}$ and therefore no contract at all in $X' \cup \{x\}$. This means that x must still be chosen when z becomes unavailable.

It follows that C_m satisfies bilateral substitutability. □

Kominers and Sönmez (2016, Proof of Lemma 1) present an alternative proof of Proposition 4.1 in the context of their model with slot-specific priorities. Their model differs in two main aspects from the worker-manager problem. First, in their model, workers do not apply for specific jobs. They apply in general for a job under a certain manager. One worker can apply to the same manager with several contracts. These contracts are distinct according to their terms, for example full-time or part-time employment. However, these terms are not necessarily specific enough to determine exactly which job the worker would fill if he were to accept a certain contract. Each priority order of a manager can rank all these contracts, which means that a manager can rank several contracts naming the same worker for a certain job as acceptable. In the worker-manager problem, each priority order can at most rank one contract for a given worker as acceptable. This is because in our setup, contracts display a much higher degree of specificity. This implies that a worker knows exactly which job he will fill if he accepts the contract. Second, the model of Kominers and Sönmez does not specify the exact number of possible contracts between a manager and a worker. In our setup, the set of possible contracts is constrained by the number of jobs that the manager oversees. These differences make the worker-manager problem more specific than Kominers and Sönmez's setup. Consequently, our proof of Proposition 4.1 is less involving than the one presented by Kominers and Sönmez.

Condition 5: Law of Aggregate Demand

Definition 4.6. *(Kominers and Sönmez, 2016) A choice function C_m satisfies the* ***law of aggregate demand*** *, if*

$$X'' \subseteq X' \Longrightarrow |C_m(X'')| \leq |C_m(X')|.$$

In words, as the choice set shrinks, the chosen set also becomes smaller or remains the same.

Claim 4. *C_m does not satisfy the law of aggregate demand*

Proof of Claim 4. We prove by example.

Consider manager m_2 from Example 3.3. Let $X'' = \{(w_3, j_1^{m_2}), (w_2, j_2^{m_2})\}$ and $X' = \{(w_3, j_1^{m_2}), (w_2, j_2^{m_2}), (w_2, j_1^{m_2})\}$. Note that the only difference between the two sets is the additional contract for worker w_2 in X'. Given these sets, we can use the manager's choice rule to determine her chosen sets:

$$\begin{aligned} C_{m_2}(X'') = &\quad \{(w_3, j_1^{m_2}), (w_2, j_2^{m_2})\} \\ C_{m_2}(X') = &\quad \{(w_2, j_1^{m_2})\} \end{aligned}$$

It follows that $|C_{m_2}(X'')| = 2 \geq 1 = |C_{m_2}(X')|$ even though $X'' \subset X'$. Hence, C_m does not satisfy the law of aggregate demand. □

Condition 6: Pareto Separability

Definition 4.7. *(Schlegel, 2015) A choice function C_m is* ***pareto separable*** *if for each $Y, Z \subseteq X$ and for all $x, x' \in X$ with $x \neq x'$ and $x_W = x'_W$:*

$$x \in C_m(Y \cup \{x, x'\}) \implies x' \notin C_m(Z \cup \{x, x'\}).$$

In words, let x and y be two contracts between manager m and worker w. If m prefers an allocation $A \cup \{x\}$ to an allocation $A \cup \{y\}$, then it never prefers another allocation $B \cup \{y\}$ to an allocation $B \cup \{x\}$.

Claim 5. *C_m is not pareto separable.*

Proof of Claim 5. We prove by example.

Again consider manager m_2 from Example 3.3. Let $x = (w_3, j_1^{m_2})$, $x' = (w_3, j_2^{m_2})$ and $Y = (w_2, j_2^{m_2})$, $Z = (w_2, j_1^{m_2})$. From the choice rule it follows that:

$$C_{m_2}(Y \cup \{x, x'\}) = \{(w_3, j_1^{m_2}), (w_2, j_2^{m_2})\}$$
$$C_{m_2}(Z \cup \{x, x'\}) = \{(w_2, j_1^{m_2}), (w_3, j_2^{m_2})\}$$

We have $x \in C_m(Y \cup \{x, x'\})$ and $x' \in C_m(Z \cup \{x, x'\})$. Hence, C_m does not satisfy pareto separability. ☐

This concludes our review of conditions for the existence of stable allocations.

4.3 EXISTENCE OF STABLE ALLOCATIONS

The proof for the existence of stable allocations in the worker-manager problem is based on an algorithm. The algorithm is called the *cumulative offer algorithm (COA)* and it processes the preference rankings of workers and managers to assign workers to jobs. In this section we first introduce the COA, as presented by Kominers & Sönmez (2016) and show that in the worker-manager problem, its outcome must always be an allocation. We then proceed to show that this allocation must be stable and by that prove that a stable allocation always exists in the worker-manager problem. We begin by introducing the COA.

In the COA, workers apply sequentially for their most preferred job by proposing a contract to the responsible manager. A manager accumulates all proposed contracts and picks those she likes most (and rejects all others). Workers whose contracts were rejected by a manager can propose again for a different job. This procedure is repeated until all workers have a job or were rejected for all of their acceptable jobs. The detailed steps are as follows:

THE CUMULATIVE OFFER ALGORITHM:

STEP 1: Some worker $w^1 \in W$ proposes his most preferred contract, $x^1 \in X(w^1)$. Manager x^1_M holds x^1 if $x^1 \in C_{x^1_M}(\{x^1\})$ and rejects x^1 otherwise. Set $A^2_{x^1_M} = \{x^1\}$ and set $A^2_{m'} = \varnothing$ for each $m' \neq x^1_M$; these are the sets of contracts available to mangers at the beginning of step 2.

STEP l: Some worker $w^i \in W$ for whom no contract is currently held by any manager proposes his most preferred contract that has not yet been rejected, $x^l \in X(w^l)$. Manager x^l_M holds the contracts in $C_{x^l_M}(A^l_{x^l_M} \cup \{x^l\})$ and rejects all other contracts in $A^l_{x^l_M} \cup \{x^l\}$; managers $m' \neq x^l_M$ continue to hold all contracts they held at the end of step $l-1$. Set $A^{l+1}_{x^l_M} = A^l_{x^l_M} \cup \{x^l\}$ and set $A^{l+1}_{m'} = A^l_{m'}$ for each $m' \neq x^l_M$.

STOP: The algorithm terminates if at any step no worker proposes a new contract. The outcome of the COA is the set of contracts held by the managers at the end of the last step before the algorithm ends.

Since there are a finite number of contracts and each contract can be offered at most one time, the COA must end eventually. Note that the order in which workers propose is not specified. This is because the outcome of the COA is independent of who proposes first, as long as the managers' choice rules satisfy bilaterally substitutability and IRC (Hirata and Kasuya, 2014). Below we present an example of the COA.

Example 4.2. Recall the preferences of the workers $W = \{w_1, w_2, w_3, w_4\}$ in Example 3.2 and the preferences of the managers $M = \{m_1, m_2\}$ in Example 3.3. Using those preferences, we run the COA. Without loss of generality, let workers propose in the order: w_1 first, followed by w_2, then w_3 and finally w_4. In step 1, worker w_1 proposes the contract $(w_1, j_2^{m_2})$ to manager m_2. Manager m_2 holds this contract, since it is acceptable to him and he currently has no other contract for that job. The sets of contracts available to managers m_1 and m_2 for step 2 are defined as $A^2_{m_1}$ and $A^2_{m_2}$. The next steps are summarized in Table 4.2. The outcome of the cumulative offer algorithm is:

$$A = \{(w_1, j_2^{m_2}), (w_2, j_1^{m_1}), (w_3, j_3^{m_1}), (w_4, j_2^{m_1})\}.$$

We now show that the outcome of the COA is, in fact, an allocation. Following Aygün & Sönmez (2012, Proof of Theorem 1) we do this by proving the Lemma below:

Lemma 4.2. *For any $m \in M$, $z \in X$ with $z(m) = m$ and $t \geq 2$,*

$$z \in A_m^{t-1} \setminus C_m(A_m^{t-1}) \text{ and } z(w) \notin \left[C_m(A_m^{t-1})\right]_W \Longrightarrow z \notin C_m(A_m^t).$$

In words, Lemma 4.2 states that if the contract z was not chosen in the previous step and manager m does not choose any contract with the worker named in z, then the contract z will also not be chosen in the current step.

Example: Cumulative offer algorithm

Step	Worker	proposes contract	to manager	The contract is	because	Set of available contracts to managers for the next step
1	w_1	$(w_1, j_2^{m_2})$	m_2.	held	it is acceptable and no other contract is held for the job.	$A^2_{m_1} = \{\varnothing\}$ $A^2_{m_2} = \Big\{\boxed{(w_1, j_2^{m_2})}\Big\}$
2	w_2	$(w_2, j_2^{m_2})$	m_2.	rejected	the manager prefers the currently held contract for that job.	$A^3_{m_1} = \{\varnothing\}$ $A^3_{m_2} = \Big\{\boxed{(w_1, j_2^{m_2})}, (w_2, j_2^{m_2})\Big\}$
3	w_3	$(w_3, j_1^{m_1})$	m_1.	held	it is acceptable and no other contract is held for the job.	$A^4_{m_1} = \Big\{\boxed{(w_3, j_1^{m_1})}\Big\}$ $A^4_{m_2} = \Big\{\boxed{(w_1, j_2^{m_2})}, (w_2, j_2^{m_2})\Big\}$
4	w_4	$(w_4, j_1^{m_2})$	m_2.	held	it is acceptable and no other contract is held for the job.	$A^5_{m_1} = \Big\{\boxed{(w_3, j_1^{m_1})}\Big\}$ $A^5_{m_2} = \Big\{\boxed{(w_1, j_2^{m_2})}, (w_2, j_2^{m_2}), \boxed{(w_4, j_1^{m_2})}\Big\}$
5	w_2	$(w_2, j_1^{m_1})$	m_1.	held	it is more preferred than the currently held contract for that job.	$A^6_{m_1} = \Big\{(w_3, j_1^{m_1}), \boxed{(w_2, j_1^{m_1})}\Big\}$ $A^6_{m_2} = \Big\{\boxed{(w_1, j_2^{m_2})}, (w_2, j_2^{m_2}), \boxed{(w_4, j_1^{m_2})}\Big\}$
6	w_3	$(w_3, j_2^{m_2})$	m_2.	rejected	the manager prefers the currently held contract for that job.	$A^7_{m_1} = \Big\{(w_3, j_1^{m_1}), \boxed{(w_2, j_1^{m_1})}\Big\}$ $A^7_{m_2} = \Big\{\boxed{(w_1, j_2^{m_2})}, (w_2, j_2^{m_2}), \boxed{(w_4, j_1^{m_2})}, (w_3, j_2^{m_2})\Big\}$
7	w_3	$(w_3, j_3^{m_2})$	m_2.	held	it is acceptable and no other contract is held for the job.	$A^8_{m_1} = \Big\{(w_3, j_1^{m_1}), \boxed{(w_2, j_1^{m_1})}\Big\}$ $A^8_{m_2} = \Big\{\boxed{(w_1, j_2^{m_2})}, (w_2, j_2^{m_2}), \boxed{(w_4, j_1^{m_2})}, (w_3, j_2^{m_2}), \boxed{(w_3, j_3^{m_2})}\Big\}$

$\boxed{}$ Contract available to the manager at the beginning of the next step.

Table 4.1: Example of the cumulative offer algorithm.

Why this implies that the outcome of the COA must be an allocation can be understood intuitively. If the COA where to end after the first step $t = 1$, the outcome must naturally be an allocation because only one contract has been proposed. For $t \geq 2$, it is not as simple. Imagine that a manager rejected several contracts of the same worker before $t-1$. If the manager decides to hold a contract in $t-1$ of that worker, this worker will, by the mechanics of the COA, not offer any other contract to any manager in t. However, if the manger does not hold a contract with the worker in $t-1$, it is possible that this worker proposes a new contract to the manager. In this case, the manager could behave in two ways that would lead to an unfeasible allocation. First, the manager chooses the newly proposed contract and a previously rejected contract with the same worker. In this case the worker would have two contracts, which is not possible under an allocation. Second, the manager chooses the new contract and a previously rejected contract involving another worker. If this other worker already holds a contract with another manager, he would be mentioned in two contracts simultaneously. This would also lead to an unfeasible allocation.

Lemma 4.2 states that none of these cases can happen and hence implies that the outcome of the COA must always be an allocation. We now proceed to show below that Lemma 4.2 is true when IRC and bilateral substitutes are satisfied by the choice rule of the managers.

Proof of Lemma 4.2. There are three cases to consider.

CASE 1: Manager m receives no offers at step t. In this case $A_m^{t-1} = A_m^t$. This immediately follows from the definition of the COP.

CASE 2: Manager m receives an offer z' from worker z_W at step t.

Since $z \in A_m^{t-1}$, we have $z' \neq z$ and thus $A_m^t = A_m^{t-1} \cup \{z'\}$. By assumption of Lemma 4.2 we have $z \notin C_m(A_m^{t-1})$. Towards a contradiction, suppose $z \in C_m(A_m^t)$. Then, by construction of our choice rule C_m, the manager m cannot have two contracts naming the same worker in her chosen set, so $z' \notin C_m(A_m^t)$. Since the only difference between A_m^{t-1} and A_m^t is the contract z', and z' is rejected at t, IRC implies that the chosen sets must stay the same, that is $C_m(A_m^t) = C_m(A_m^{t-1})$. This contradicts $z \in C_m(A_m^t) \setminus C_m(A_m^{t-1})$ and completes Case 2.

CASE 3: Manager m receives an offer x from worker $x_W \neq z_W$ at step t.

Let $Y = A_m^{t-1} \setminus \{y \in X \mid y_W \in \{x_W, z_W\}\}$. Observe that $x_W, z_W \notin Y_W$. Since worker x_W makes an offer at step t, we have $x_W \notin [C_m(A_m^{t-1})]_W$; furthermore, by assumption of Lemma 4.2 $z_W \notin [C_m(A_m^{t-1})]_W$. Finally by IRC, $C_m(A_m^{t-1}) = C_m(Y \cup \{z\})$, and therefore $z \notin C_m(Y \cup \{z\})$, which in turn implies

$$z \notin C_m(Y \cup \{x, z\}) \tag{4.2}$$

by bilateral substitutability. Towards a contradiction, suppose $z \in C_m(A_m^t)$. Since $z \notin C_m(A_m^{t-1})$, that means $C_m(A_m^t) \neq C_m(A_m^{t-1})$. This in turn implies $x \in C_m(A_m^t)$ by IRC and the fact that $A_m^t = A_m^{t-1} \cup \{x\}$. Thus, $x, z \in C_m(A_m^t)$ which means neither worker x_W nor worker z_W can have other contracts in $C_m(A_m^t)$. Therefore, IRC implies $x, z \in C_m(Y \cup \{x, z\})$, contradicting relation (4.2) and completing Case 3. This completes the proof of the Lemma 4.2. □

We have shown that the outcome of the COA is always an allocation. We now proceed to prove that there always exists a stable allocation in the worker-manager problem by showing that the outcome of the COA must be stable. This approach is based on Aygün and Sönmez (2012, proof of Theorem 1).

Theorem 4.3. *A stable allocation always exists in the worker-manager problem.*

Proof of Theorem 4.3. Let X' be the outcome of the COA with a total of T steps. From Lemma 4.2 we know that X' is an allocation. We now show that X' is stable by proving that X' is individually rational and unblocked.

X' is individually rational:

First, observe that no worker can block X' since a worker never offers an unacceptable contract. Hence $C_W(X') = X'$. Turning to the managers, suppose $C_M(X') \neq X'$, and observe that $C_M(X') = \bigcup_{m \in M} C_m(A_m^T)$ under IRC. Hence there exists a manager m and a contract x such that $x \in C_m(A_m^T)$ but $x \notin C_m(C_m(A_m^T))$. This is ruled out by IRC and therefore $C_M(X') = X'$.

X' is unblocked:

Towards a contradiction, suppose there exists a manager m and a set of contracts $X'' \notin C_m(X')$ such that

$$X'' = C_m(X' \cup X'') \text{ and } X''(w) = C_w(X' \cup X'') \ \forall w \in X''(w)$$

Remember that we defined $X'(m) \equiv \{x \in X' \mid x_M = m\}$. That is, $X'(m)$ is the subset of X' that pertains to manager m. Observe that $X'(m) = C_m(A_m^T)$ by the mechanics of the cumulative offer algorithm. Also recall that we have already shown $C_m(X') = X'(m)$ by the above individual rationality argument. Hence,

$$X'(m) = C_m(X') = C_m(A_m^T) \tag{4.3}$$

Since $X'' = C_m(X' \cup X'')$, we have $x_M = m$ for all $x \in X''$. Moreover let $x \in X''$, $x' \in X'$, $x \neq x'$ and $x'_W = x_W$. Since $X''(w) = C_w(X' \cup X'') \ \forall w \in X''(w)$,

$$x \ \ P_{x_W} \ \ x'$$

Therefore, each contract in X'' is offered to manager m by step T by the mechanics of the COA. Hence,

$$X'' \subseteq A_m^T \tag{4.4}$$

This in turn implies

$$X'' = C_m(X' \cup X'') = C_m(X'(m) \cup X'') = C_m(C_m(X') \cup X'') = C_m(A_m^T) = C_m(X')$$

contradicting $X'' \neq C_m(X')$.

Here,

1. the first equality holds by assumption;
2. the second equality holds by IRC since none of the contracts in $X' \setminus X'(m)$ pertain to manager m, and as such they are automatically rejected by m;
3. the third equality holds by the Relation (4.3);
4. the fourth equality holds by IRC together with Relations 4.3 and 4.4 since $C_m(X') \cup X'' \subseteq A_m^T$ and only the rejected contracts are removed between A_m^T and $(C_m(X') \cup X'')$; and
5. the last equality holds by the Relation 4.3.

This shows that X' is stable, completing the proof. □

In this chapter we have shown that a stable allocation always exists in the worker-manager problem. However, stability is not everything in our setting. There is another notion that is central to capturing the essence of the rotation exercise at the GSC, which we investigate in the next chapter: individual rationality with respect to the initial allocation.

5

INDIVIDUAL RATIONALITY WITH RESPECT TO THE INITIAL ALLOCATION

We now look at a notion that stems from the fact that workers in the worker-manager problem have an initial allocation. This initial allocation implies that workers distinguish between two types of acceptable contracts: the ones they find better than their initial contract and the ones they find worse. This distinction is crucial when it comes to understanding a worker's satisfaction with a contract he is assigned to when he has to switch jobs. The classical definition of stability does not take this difference into account, and hence does not allow for a complete analysis of the worker-manager problem.

In this chapter, we first define the notion of *individual rationality with respect to the initial allocation*. Second, we show that the COA can be modified in such a way that it will always find an allocation which satisfies this new notion. Finally, we relate the new notion to the stability notion from the previous chapter.

5.1 DEFINITION

Definition 5.1. *An allocation A is* ***individually rational with respect to the initial allocation*** *if for all $w \in W$ we have:*

$$C_w(A) \; R_w \; A^0(w).$$

In words, individual rationality with respect to the initial allocation requires that, given an allocation A and the initial allocation A^0, no worker is assigned a contract that she finds worse than her initial contract.

5.2 THE CUMULATIVE OFFER ALGORITHM WITH ADJUSTED PREFERENCES

In order to find allocations that respect individual rationality with respect to the initial allocation, we modify the inputs to the COA. We adjust the jobs' true priority orders such that an initial contract is placed at the top of the list of its respective job, i.e. the initial contract is the contract with the highest priority for each job. Then we apply the COA as defined in Chapter 4. The outcome must be individually rational with respect to the initial allocation because in the case that a worker is not accepted to a contract that he prefers over his initial contract, the adjusted job priorities guarantee that this worker can always hold on to his initial contract and so avoid being assigned to a contract that he finds worse.

5.3 COMPARISON TO STABILITY

Recall from Chapter 4 that when an allocation is stable, no participant has the ability nor the incentive to disrupt it by deviating from their assigned contract. However, stability does not capture the notion of individual rationality with respect to the initial allocation, which is central in the worker-manager problem. In order to analyze the worker-manager problem, it is useful to compare the components of stability to the notion of individual rationality with respect to the initial allocation.

Individual rationality with respect to the initial allocation, which is only defined for workers, is stricter than individual rationality, the first condition for stability, because we assume that a worker always prefers the initial contract to the empty set. Thus, when individual rationality with respect to the initial allocation is satisfied, so is individual rationality for the workers. The second condition for stability, namely that the allocation is unblocked, might be violated because of the adjustments we impose on the managers' preference rankings. The following example illustrates this problem. Note that this example is independent from our main example.

Example 5.1. Assume there are two workers $\{w_1, w_2\}$ and two managers with one job each $J = \{j_1^{m_1}, j_1^{m_2}\}$ and the initial allocation $A = \{(w_1, j_1^{m_1})^0, (w_2, j_1^{m_2})^0\}$. The workers' preferences and the jobs' priorities are given by:

P_{w_1}	P_{w_2}	$\Pi_{j_1^m}$	$\Pi_{j_2^m}$
$(w_1, j_1^{m_1})^0$	$(w_2, j_1^{m_1})$	$(w_2, j_1^{m_1})$	$(w_1, j_1^{m_2})$
$(w_1, j_1^{m_2})$	$(w_2, j_1^{m_2})^0$	$(w_1, j_1^{m_1})^0$	$(w_2, j_1^{m_2})^0$
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$

The adjusted job priorities would be:

$\Pi^{adjusted}_{j_1^{m_1}}$	$\Pi^{adjusted}_{j_1^{m_2}}$
$(w_1, j_1^{m_1})^0$	$(w_2, j_1^{m_2})^0$
$(w_2, j_1^{m_1})$	$(w_1, j_1^{m_2})$
$\varnothing$	$\varnothing$

The outcome of the COA with adjusted preferences is:

$$A' = \{(w_1, j_1^{m_1})^0, (w_2, j_1^{m_2})^0\}.$$

However, this allocation is not stable since worker w_2 and the manager overseeing job $j_1^{m_2}$ would rather sign contract $(w_2, j_1^{m_1})$. Hence, they block the outcome allocation of the COA with adjusted preferences with regard to the 'original' stability definition.

In the literature, blocking pairs are seen as a problem because they have the ability to disrupt an allocation. This is not the case in the worker-manager problem because contracts within an institution are enforced by an authority. It is not possible for managers and workers to switch their contracts for other contracts on their own. However, blocking pairs are still problematic. The fact that the workers involved in a blocking pair cannot deviate from their assigned contract might undermine the trust in the enforcing authority. Nonetheless, it is important to note that if the COA with adjusted preferences is used to find an allocation, the preference orders of the managers are still completely respected with the exception of their initial contracts. This means that the blocking pairs in such an allocation must always include a manager who was forced to accept his initial worker. We will use this insight for our later analysis.

To sum up, we have introduced two concepts, stability and individual rationality with respect to the initial allocation, that allow us to analyze allocations from different perspectives. These concepts will be useful when we explore the topic of the next chapter: mechanisms.

6

MECHANISMS

Previously, we modelled agents' preferences and analyzed allocations and their properties in relation to those preferences. Now we want to determine how to get from preferences to a desirable allocation, that means an allocation which satisfies certain properties. To do that we introduce the concept of 'mechanisms'. Intuitively, a mechanism consists of three parts: an *input* to a *black box* which determines an *output*. In our case, inputs will simply be the preference orders for the workers and the priority rankings for the managers. The output will be an allocation. We will go into more detail about the black box later in this chapter. We now proceed to define mechanisms formally and present their properties.

6.1 MECHANISMS

Definition 6.1. *(Sönmez and Switzer, 2013)*

*A **mechanism** in the worker-manager problem is:*

- *a strategy space S_i for $i \in W \times M$, and*
- *an outcome function $\varphi : \times_{i \in W \times M} S_i \to \mathscr{A}$ that selects an allocation for each choice of workers' and managers' strategies.*

In words, a strategy space defines which information the workers and the managers can submit. The submissions of all participants are the input to the mechanism. The outcome function defines an allocation (the outcome), based on the strategies (the input). Given the strategy vector s and the outcome function φ, we denote the resulting allocation as $\varphi(s)$. This outcome function corresponds to the 'black box' we mentioned before. The black box maps the inputs to the outputs. Formally, it is a function. Note that this black box does not define *how* we determine an allocation. It only defines *which* allocation is chosen given certain inputs. For any real world application we would need to know exactly how to calculate the outcome (allocation). But for the formal analysis of a mechanism, it is enough to know what the inputs and respective outputs are.

Example 6.1. Recall the COA from Chapter 4. This algorithm defines step by step how to get from some input preferences to an allocation. If we want to define the corresponding *'cumulative offer mechanism'*, we need to define the strategy spaces and the outcome function. The strategy spaces would simply be all possible preferences for workers and priority rankings for managers, because those are the types of information that the COA processes. The COA would not be able to process pieces of information like "it is Wednesday". The outcome function of the cumulative offer mechanism would be the function that picks exactly the allocation that results by running the COA with respect to submitted preferences and priority rankings. Note that to define this outcome function, we do not need to know the actual steps of the COA. We just need to know what outcome allocation the COA produces.

6.2 DIRECT MECHANISMS

Mechanisms can be very complex. There could be multiple rounds of submissions where each round could depend on what happened in the last round. A mechanism could even allow people to submit information that is not relevant for finding the outcome. However, there is a subclass of mechanisms which is much easier to analyze: direct mechanisms.

Direct mechanisms are mechanisms that impose restrictions on the strategy spaces, namely that each participant may only once submit a preference ranking and all submissions must occur simultaneously (Sönmez and Switzer, 2013). Direct mechanisms are useful to simplify the analysis of complex real-world allocation problems. They are also crucial for a seminal insight of mechanism design: the revelation principle, which we look at in the following section.

6.3 SOCIAL CHOICE FUNCTIONS AND THE REVELATION PRINCIPLE

To explain the revelation principle, we must first define the concept of social choice functions. **Social choice functions (SCF)** are similar to mechanisms in that they also map preferences to an outcome. However, whereas a mechanism defines what happens given *reported* preferences, a SCF defines what would happen if we knew the *true* preferences of all participants. Note that reported preferences do not necessarily correspond to the true preferences of a participant. A participant could submit a preference order to a mechanism that does not represent his true preferences. In a sense he can submit a lie. There is no way of telling if a submitted preference order is a participants' true preference order or not. Since a SCF defines what would happen if we knew all true preferences of all participants, it can be thought of as a benchmark for what a mechanism should achieve. Having explained what a SCF is, we can now turn to *the revelation principle* as defined by Nisan et al. (2007, p.224):

Definition 6.2 (The revelation principle)**.** *If a social choice function can be implemented by an arbitrary mechanism, then the same SCF can be implemented by a direct mechanism with the same equilibrium outcome and where participants submit their true preferences.*

This implies that when searching for an optimal mechanism, it is enough to restrict the search to direct mechanisms only, making it easier to find a useful mechanism for a given problem. We use this approach to find improvements to the GSC rotation exercise.

Having defined what a mechanism is, we can now define properties of mechanisms.

6.4 MECHANISM PROPERTIES

We describe three properties of mechanisms.

6.4.1 *Stability*

In Chapter 4 we defined stable allocations as allocations that are individually rational and unblocked. Using this definition, we say that:

Definition 6.3. *A mechanism is* ***stable*** *if it always selects an outcome that is stable with respect to the submitted preferences of the workers and the submitted job priority orders of the managers.*

6.4.2 *Individual rationality with respect to the initial allocation*

Similarly, we can transfer the central notion of Chapter 5 to mechanisms and say that:

Definition 6.4. *A mechanism is* ***individually rational with respect to the initial allocation*** *if it always selects an outcome where every worker is assigned a contract that she finds at least as good as her initial contract.*

6.4.3 *Strategy-proofness*

The first two mechanism properties concerned only the outcome of a mechanism. We now turn to the incentives that a mechanism creates. Thus we introduce the concept of strategy-proofness.

Definition 6.5. *(Sönmez and Switzer, 2013) A mechanism is* ***strategy-proof for the workers*** *if there is no worker $w \in W$, preference profile $P_W \in \mathcal{P}_W$ and $P_w, \bar{P}_w \in \mathcal{P}_w$ with $P_w \neq \bar{P}_w$ such that:*

$$\varphi(\bar{P}_w, P_{-w}) \ P_w \ \varphi(P_w, P_{-w})$$

In words, a mechanism is strategy-proof if submitting the true preference is a dominant strategy for all workers. Put differently, if a mechanism is strategy-proof for the workers, each worker w would want to submit his true preferences rather than lying, no matter what all the other workers do. In a mechanism that is not strategy-proof for workers, workers might be able to improve their outcome allocation by submitting preference rankings different from their true ones, that is, they need to strategize. This is problematic. A seminal example illustrating why a mechanism that is not strategy-proof can be detrimental is that of the Boston mechanism.

The Boston mechanism was used to allocate children to public schools in Boston. Similar to the procedures we have mentioned before, it relied on parents submitting ordered rankings of their most preferred schools and schools submitting priority lists of students to a centralized clearinghouse. These were then used to assign students to the schools (Ergin and Sönmez, 2006). However, the mechanism did not encourage parents to submit their true preferences. As Pathak and Sönmez (2008, p.1636) describe it, "the Boston mechanism attempts to assign as many students as possible to their first choice school, and only after all such assignments have been made does it consider assignments of students to their second choices, and so on. If a student is not admitted to her first choice school, her second choice may be filled with students who have listed it as their first choice. That is, a student may fail to get a place in her second choice school that would have been available had she listed that school as her first choice. If a student is willing to take a risk with her first choice, then she should be careful to rank a second choice that she has a chance of obtaining." Pathak and Sönmez found that students with 'sophisticated' parents who understood how to strategize correctly, had an advantage over the students with 'sincere' parents who submitted their true preferences. Pathak and Sönmez argued that introducing a strategy-proof mechanism would avoid this imbalance. Additionally, a strategy-proof mechanism would save parents the time and energy they would need to identify how to behave optimally.

6.4.4 *Comparison of stability and strategy-proofness*

While stability and strategy-proofness are both desirable properties for mechanisms, it is impossible to find a mechanism that satisfies both. This insight is known as the *impossibility theorem* (Roth and Sotomayor, 1990, Thm. 4.4).

Theorem 6.1 (Impossibility Theorem)**.** *No stable matching mechanism exists for which stating the true preferences is a dominant strategy for every participant.*

However, it is possible to find a stable mechanism for which stating the true preferences is a dominant strategy for one group of participants. We will use these insights later to construct and examine different mechanisms. In the next chapter we will analyze the currently employed mechanism of the job rotation in the GSC.

7

GSC MECHANISM

We now turn to the rotation exercise currently employed at the GSC. In the first section of this chapter we formally present the rotation exercise, which from now on we will refer to as the *GSC process*. In the second section we go through a detailed example. In the third and fourth sections we define and analyze the corresponding *GSC mechanism*, respectively. In the final section we look at the results from the pilot rotation run between 2017 and 2018.

7.1 THE GSC PROCESS

The GSC process is centrally organised by the HR department of the GSC and consists of multiple *rounds* (2018d). Each round consists of three *phases*. Before we go into detail, an overview of the three phases is presented below:

- PHASE 1: Workers and managers submit preferences over contracts.
- PHASE 2: Contracts of mutual interest get a priority rank using a predefined rule.
- PHASE 3: Allocation contracts are chosen following the *'GSC Algorithm'*. Assigned workers and jobs are taken out of the process. Unassigned workers and jobs go to the next round.

The GSC process is repeated for as many rounds as necessary until all workers are assigned a contract.We now look at each phase in more detail.

Phase 1: Preference Submission

Each worker submits one ranking over available contracts. The HR Department stipulates the minimum number, greater than zero, of contracts to be ranked. Currently, the minimum number is five. Each manager submits a priority rank of contracts for each job he oversees. Managers must also rank a stipulated minimum number of contracts for each of their jobs. Currently, the minimum number is three (2018d). Note that in the GSC process, participants cannot submit their true preferences, since they are not allowed to rank their initial contract nor the null contract.

Phase 2: The Grid

A certain priority is given to each contract in which there was a display of mutual interest, that is, when a worker and a manager included the same contract in their submitted rankings. We denote this set of contracts of mutual interest by $\tilde{X}$. The priority given to a contract in $\tilde{X}$ depends on which rank the worker's input preference and the manager's input priority order gave that contract. The exact priority assigned to a contract is defined by a rule called *the grid* (2017i; 2018d). An excerpt of the currently used grid can be seen in Figure 7.1 below. The entire grid can be found in appendix A.4.

Priority	Rank on P_w	Rank on Π_j
1	1	1
2	1	2
3	2	1
4	1	3
5	2	2
6	3	1
7	1	4
8	2	3
9	3	2
10	4	1
11	1	5
12	2	4
⋮	⋮	⋮

Table 7.1: Excerpt of the currently used grid

Example 7.1. Using the grid in Figure 7.1, if both a worker and a manager ranked the same contract as their first choice, this potential contract would be given the priority 1. If a contract was a worker's first choice, but the manager ranked the contract as her second choice, this potential contract will be given a priority 2.

It is important to note that there can be multiple contracts with the same priority . However, contracts that have the same priority will never name the same worker nor job. Formally, for two contracts $x, x' \in \tilde{X}$ with the same priority it must be that $x_W \neq x'_W$ and $x_J \neq x'_J$.

Although the grid might seem arbitrary at first, there is a system behind its construction. Observe that the sums of the ranks of the workers and managers (columns two and three of Figure 7.1) in the same row (weakly) increase as the priority increases (as you move downwards in the figure). Whenever the sum is the same between rows, the contract where the rank given by the worker is lower, is listed first. In this sense, workers' preferences are given precedence over managers' preferences. Example 7.2 offers clarification. Constructing a grid like this is an intuitive way of taking the preferences of all participants into account. This is why this approach has been widely implemented. For example, very similar grids were used to match young doctors to hospitals in cities across the UK like Newcastle, Birmingham and Edinburgh in the 1960s (Roth, 1991). The Boston mechanism which we mentioned in Chapter 6 was also based on such a grid (Ergin and Sönmez, 2006).

Example 7.2. In the excerpt from the grid below, the sum of the 'Rank on P_w' and 'Rank on Π_j' in the first row is 2, while the sum of those ranks in the second row is 3. The sums of the 'Rank on P_w' and 'Rank on Π_j' of rows 2 and 3 are both 3, but priority is given to the 'Rank on P_w', that is, 1:2 is prioritized over 2:1.

Priority	Rank on P_w		Rank on Π_j		Sum
1	1	+	1	=	2
2	1	+	2	=	3
3	2	+	1	=	3
4	1	+	3	=	4
5	2	+	2	=	4
⋮	⋮		⋮		⋮

Phase 3: The GSC Algorithm

An algorithm, referred to as the *GSC algorithm*, determines which of the contracts in $\tilde{X}$ will be part of the outcome allocation of the GSC process. Let A^{GSC} denote this outcome allocation. In the beginning of the first round let $A^{GSC} = \varnothing$. The algorithm then proceeds as follows.

THE GSC ALGORITHM

STEP 1: If there are contracts with priority 1, add all of those contracts from $\tilde{X}$ to the set A^{GSC}. Then remove all contracts from $\tilde{X}$ which name a worker or a job that is named in a contract in A^{GSC}. Then continue with step 2. If there are no contracts with priority 1 in the beginning of step 1, continue directly with step 2.

STEP k: If there are contracts with priority k, add all of those contracts from $\tilde{X}$ to the set A^{GSC}. Then remove all contracts from $\tilde{X}$ which name a worker or a job that is named in a contract in A^{GSC}. Then continue with step $k+1$. If there are no contracts with priority k in the beginning of step k, continue directly with step $k+1$.

STOP: The GSC algorithm terminates when there are no more contracts left in $\tilde{X}$.

If some workers remain unmatched, we proceed with another round. This means that unassigned workers and managers submit new preferences and priority orders, respectively, over unfilled jobs and unmatched workers. $\tilde{X}$ is redefined as all new available contracts and the GSC algorithm is repeated.

As soon as all workers are assigned to a job, the entire GSC process ends. The process must end eventually because in each round at least one worker is assigned to a contract. This directly follows from the stipulated minimum number of contracts that must be ranked. Since we have a finite number of workers, the process must end. The final allocation A^{GSC} is the outcome of the GSC process.

This concludes the description of the GSC process. We now present a detailed example.

7.2 EXAMPLE OF THE GSC PROCESS

This example illustrates the GSC process. Recall the setup of the main example we developed over the last chapters. We have four workers $W = \{w_1, w_2, w_3, w_4\}$, two managers $M = \{m_1, m_2\}$ with $J = \{j_1^{m_1},\ j_2^{m_1}\} \cup \{j_1^{m_2},\ j_2^{m_2},\ j_3^{m_2}\}$ and the initial allocation:

$$A^0 = \left\{ \left(w_1, j_1^{m_1}\right)^0,\ \left(w_2, j_2^{m_1}\right)^0,\ \left(w_3, j_1^{m_2}\right)^0,\ \left(w_4, j_2^{m_2}\right)^0 \right\}.$$

<u>Round 1</u>

PHASE 1: PREFERENCE SUBMISSION

Assume that the workers have the same true preferences as in Example 3.2 and the managers as in Example 3.3. For this example, the minimum number of contracts that workers must rank is three. Managers must rank at least one contract for each job they oversee. Recall that participants cannot submit their true preferences. For this example, assume the following submitted preferences.

P_{w_1}	P_{w_2}	P_{w_3}	P_{w_4}
$(w_1, j_2^{m_2})$	$(w_2, j_2^{m_2})$	$(w_3, j_1^{m_1})$	$(w_4, j_1^{m_2})$
$(w_1, j_3^{m_2})$	$(w_2, j_1^{m_1})$	$(w_3, j_2^{m_2})$	$(w_4, j_3^{m_2})$
$(w_1, j_1^{m_2})$	$(w_2, j_3^{m_2})$	$(w_3, j_3^{m_2})$	$(w_4, j_1^{m_1})$
		$(w_3, j_2^{m_1})$	

Figure 7.1: Workers' submitted preferences in round 1

$\Pi_{j_1^{m_1}}$	$\Pi_{j_2^{m_1}}$	$\Pi_{j_1^{m_2}}$	$\Pi_{j_2^{m_2}}$	$\Pi_{j_3^{m_2}}$
$(w_2, j_1^{m_1})$	$(w_3, j_2^{m_1})$	$(w_1, j_1^{m_2})$	$(w_1, j_2^{m_2})$	$(w_2, j_3^{m_2})$
$(w_3, j_1^{m_1})$			$(w_2, j_2^{m_2})$	$(w_1, j_3^{m_2})$
$(w_4, j_1^{m_1})$			$(w_3, j_2^{m_2})$	$(w_4, j_3^{m_2})$
				$(w_3, j_3^{m_2})$

Figure 7.2: Managers' submitted preferences in round 1

PHASE 2: THE GRID

Figure 7.3 shows the set of possible contracts and their respective rankings by the workers and managers.

$\tilde{X}$	$j_1^{m_1}$	$j_2^{m_1}$	$j_1^{m_2}$	$j_2^{m_2}$	$j_3^{m_2}$
w_1			[3:1]	[1:1]	[2:2]
w_2	[2:1]			[1:2]	[3:1]
w_3	[1:2]	[4:1]		[2:3]	[3:4]
w_4	[3:3]		[1:-]		[2:3]

$\xrightarrow[\text{grid}]{\text{the}}$

$\tilde{X}$	$j_1^{m_1}$	$j_2^{m_1}$	$j_1^{m_2}$	$j_2^{m_2}$	$j_3^{m_2}$
w_1			[6]	[1]	[5]
w_2	[3]			[2]	[6]
w_3	[2]	[10]		[8]	[18]
w_4	[13]				[8]

Figure 7.3: Priorities of contracts of mutual interest in round 1

PHASE 3: THE GSC ALGORITHM

STEP 1: There exists one contract with priority 1, namely the contract $(w_1, j_2^{m_2})$. This contract is added to the final allocation A^{GSC}. Now, all contracts involving worker w_1 and job $j_2^{m_2}$ are unavailable for the next rounds. Graphically one can think of this as crossing out the first row and the fourth column from the matrix on the right-hand side of Figure 7.3.

STEP 2: The contract with the next highest priority is that of $(w_3, j_1^{m_1})$, with a priority of 2. w_3 and $j_1^{m_1}$ are added to the final allocation and are crossed out from the table of all possible matches. Note that there is another priority 2 contract, namely $(w_2, j_2^{m_2})$. However, the job $j_2^{m_2}$ was already assigned in step 1, hence this contract is not available anymore.

STEP 3-5: No contracts with priority 3 to 5 are available. Continue to the next step.

STEP 6: The contract with the next highest priority is that of $(w_2, j_3^{m_2})$, with a priority of 6. This contract is added to the final allocation and is crossed out from the table of all possible matches.

At this point there are no more available contracts and worker w_4 and the jobs $j_2^{m_1}$ and $j_1^{m_2}$ remain without an assigned contract. Hence we move on to round 2.

Round 2

PHASE 1: PREFERENCE SUBMISSION

The unmatched worker and the unfilled jobs must submit new rankings. Since there is only one worker left, the jobs can only rank one contract each and the worker can only rank the two jobs. We assume:

P_{w_4}	$\Pi_{j_2^{m_1}}$	$\Pi_{j_1^{m_2}}$
$(w_4, j_1^{m_2})$	$(w_4, j_2^{m_1})$	$(w_4, j_1^{m_2})$
$(w_4, j_2^{m_1})$		

Figure 7.4: Workers' and managers' submitted preferences in round 2

PHASE 2: THE GRID

The available contracts are:

$\tilde{X}$	$j_2^{m_1}$	$j_1^{m_2}$
w_4	[2:1]	[1:1]

$\xrightarrow[\text{grid}]{\text{the}}$

$\tilde{X}$	$j_2^{m_1}$	$j_1^{m_2}$
w_4	[3]	[1]

Figure 7.5: Priorities of contracts of mutual interest in round 2

PHASE 3: THE GSC ALGORITHM

STEP 1: There exists one contract with priority 1, namely the contract $(w_4, j_1^{m_2})$. This pair is added to final allocation A^{GSC}.

Since all workers are matched, this concludes the GSC process. The final allocation is: $A^{GSC} = \{(w_1, j_2^{m_2}), (w_2, j_3^{m_2}), (w_3, j_1^{m_1}), (w_4, j_1^{m_2})\}$.

7.3 THE GSC MECHANISM

We now define the GSC mechanism, which selects the allocation obtained by running the GSC process. Recall that a mechanism is composed of the strategy spaces of the workers and the managers and an outcome function.

Strategy spaces of the workers: The workers must submit rankings over contracts for each round they are part of. These rankings must at least contain the stipulated minimum number of contracts. If there are fewer contracts available, they must rank all contracts. They are not allowed to rank their initial contract.

Strategy spaces of the managers: In each round the managers must submit a priority order for each of their jobs that is not yet assigned a contract. Each priority order must at least contain the stipulated minimum number of contracts. If there are fewer contracts available for a job, the priority order of this job must contain all contracts. The job priorities must not contain their respective initial contract.

The **outcome function of the GSC mechanism** φ^{GSC} selects the outcome which is obtained by going through the GSC process.

7.4 ANALYSIS OF THE GSC MECHANISM

As mentioned before, the approach of giving priorities to contracts has been commonly used with some variations. Mechanisms that use such priority systems are called *priority matching mechanisms* (Ergin and Sönmez, 2006; Roth, 1991). As a special case of a priority matching mechanism, the GSC mechanism shares some of the properties of this family of mechanisms. We will now examine what these properties are.

7.4.1 *Stability*

Recall that a mechanism is stable if it always selects an outcome which is individually rational and unblocked. We prove that the GSC mechanism is not stable by giving an example where its outcome is blocked.

Proposition 7.1. *The GSC mechanism is not stable.*

Proof of Proposition 7.1. Assume that the outcome of the GSC mechanism is as in Section 7.2:

$$A^{GSC} = \{(w_1, j_2^{m_2}), (w_2, j_3^{m_2}), (w_3, j_1^{m_1}), (w_4, j_1^{m_2})\}.$$

Consider manager m_1. His chosen set from allocation A^{GSC} is $C_{m_1}(A^{GSC}) = \{(w_3, j_1^{m_1})\}$. Now let $B = \{(w_2, j_1^{m_1})\}$. Given the true preferences of the manager, we have that

$$C_{m_1}(A^{GSC} \cup B) = \{(w_2, j_1^{m_1})\} = B$$

Given the true preferences of worker w_2, we know that $(w_2, j_1^{m_1})\ P_{w_2}\ (w_2, j_3^{m_2})$. This means that: $B = B(w_2) = C_{w_2}(A^{GSC} \cup B)$. Hence, manager m_1 with his first job $j_1^{m_1}$ and worker w_2 block the outcome of the GSC mechanism. Thus, the GSC mechanism does not always produce a stable outcome. □

It is important to note that the GSC mechanism also violates individual rationality. For example, imagine a worker who has fewer acceptable contracts than the minimum number of contracts to be ranked, as stipulated by the HR department. He will be forced to submit some unacceptable contracts in his ranking. If the outcome of the GSC process assigned this worker to an unacceptable contract, it would violate individual rationality. Alternatively, Roth (1991, proof of Proposition 10) shows that no one-round priority matching mechanism is stable.

7.4.2 *Individual rationality with respect to the initial allocation*

Recall that individual rationality with respect to the initial allocation requires that each worker finds his outcome allocation contract at least as good as his initial contract.

Proposition 7.2. *The GSC mechanism violates individual rationality with respect to the initial allocation.*

Proof of Proposition 7.2. Assume that the outcome of the GSC mechanism is as in Section 7.2:

$$A^{GSC} = \{(w_1, j_2^{m_2}), (w_2, j_3^{m_2}), (w_3, j_1^{m_1}), (w_4, j_1^{m_2})\}.$$

Worker w_2 is employed at $j_3^{m_2}$. However, when we look at worker w_2's true preferences in Example 3.2, we see that this new job was ranked below his initial contract:

$$(w_2, j_2^{m_1})\ P_{w_2}\ (w_2, j_3^{m_2})$$

Hence, the mechanism does not respect individual rationality with respect to the initial allocation. □

7.4.3 *Strategy-proofness*

The GSC mechanism may request participants to submit several rankings over multiple rounds. It is thus difficult to find a dominant strategy and it is unclear whether such a strategy exists at all. For this reason, we do not use the formal definition of strategy-proofness presented in Chapter 6. However, it is possible to show that a worker can receive a better allocation contract by submitting an "untrue" preference order. In other words, we can show that the GSC mechanism incentivizes strategizing. This is analogous to the idea that a mechanism is not strategy-proof.

To illustrate, recall that in the example from Section 7.2 it is possible for worker w_2 to improve her assigned contract by submitting a different ranking of contracts. Assume that in her new submission, denoted by $\bar{P}_{w_2}$, the contracts $(w_2, j_1^{m_1})$ and $(w_2, j_2^{m_2})$ are switched:

$\bar{P}_{w_2}$
$(w_2, j_1^{m_1})$
$(w_2, j_2^{m_2})$
$(w_2, j_3^{m_2})$

With $\bar{P}_{w_2}$, the GSC mechanism yields a different final allocation, namely:

$$\bar{A}^{GSC} = \{(w_1, j_2^{m_2}), (w_2, j_1^{m_1}), (w_3, j_2^{m_1}), (w_4, j_3^{m_2})\}$$

In $\bar{A}^{GSC}$, worker w_2 gets her second choice instead of her third choice (as in A^{GSC}) with respect to her true preferences P_{w_2}, and is thus able to improve her assignment by lying about her preferences. It follows that the GSC mechanism incentivizes strategizing. Alternatively, Roth (1991, proof of Proposition 5) shows that one-round priority matching mechanisms are never strategy-proof for any group of participants.

This concludes our analysis of the GSC mechanism. In the next section we will take a look at the outcome of the pilot rotation exercise.

7.5 THE PILOT ROTATION OF 2017–2018

The pilot rotation exercise was launched in September 2017 and the first effective reassignment is scheduled for July 2018 (2018d). There were 14 workers in this pilot rotation exercise, all of which had to participate mandatorily, that is, there were no volunteers. There were 13 managers overseeing a total of 15 jobs, where one manager oversaw three jobs and the remaining managers oversaw one job each. Table 7.2 shows all workers' and managers' preferences submitted in the pilot and the results of the first round of matching (2018b). Three workers remained unassigned after the first round. The second round did not take place as foreseen, meaning that the GSC process was not repeated a second time. Instead, the unassigned workers found a suitable position through dialogue outside of the GSC process (2017c).

	$j_1^{m_1}$	$j_1^{m_2}$	$j_1^{m_3}$	$j_1^{m_4}$	$j_1^{m_5}$	$j_1^{m_6}$	$j_1^{m_7}$	$j_1^{m_8}$	$j_1^{m_9}$	$j_1^{m_{10}}$	$j_2^{m_{10}}$	$j_3^{m_{10}}$	$j_1^{m_{11}}$	$j_1^{m_{12}}$	$j_1^{m_{13}}$
w_1							[5:2]			[3:2]	[4:1]		**[1:1]**		[2:-]
w_2	[5:2]	**[1:1]**						[3:2]	[2:1]					[4:5]	
w_3	[1:4]							[4:-]					[3:-]	**[2:1]**	[5:-]
w_4			[2:3]	[4:2]		**[1:2]**		[3:-]						[5:4]	
w_5	[5:1]		[4:4]			[3:1]	[2:1]								**[1:1]**
w_6			[2:2]		**[1:1]**			[3:1]	[5:2]				[4:2]		
w_7			**[1:1]**	[2:3]	[3:2]				[5:3]			[4:1]			
w_8								[4:-]		**[1:1]**	[2:2]	[3:2]		[5:2]	
w_9				**[1:1]**	[2:5]		[5:-]	[3:-]	[4:-]						
w_{10}	[5:-]		[4:-]	[2:-]	[3:4]			**[1:3]**							
w_{11}	**[1:3]**		[2:-]			[4:3]		[3:-]							[5:-]
w_{12}					[1:3]			[3:-]	[2:-]				[4:3]	[5:3]	
w_{13}				[5:-]	[1:6]		[2:-]	[4:-]							[3:2]
w_{14}	[3:-]				[4:7]	[5:-]	[2:-]							[1:-]	

Allocation contracts from the first round.

Table 7.2: Rankings of the 2017 pilot participants.

Now we examine the properties of the outcome of the first round. First, we cannot tell whether the allocation that resulted from round 1 of the pilot is individually rational nor individual rational with respect to the initial allocation, since workers and managers were not allowed to rank the empty set nor their initial contract in their submitted preferences. Second, we can see that the allocation is unblocked with respect to the submitted preferences. Only one worker and three managers did not get their first choice - so these would be the only participants who might benefit from disregarding their match. The worker who did not get his first choice (w_3) only ranked two out of the three managers who did not get their first choice (m_1 and m_8). One of those managers (m_8) was his fourth choice, so he was less preferred than his match. The other manager (m_1) was actually his first choice, but that manager prefers his match (w_{11}) to the worker (w_3). Thus, there are no possibilities for any group of workers and managers to block the allocation. However, this does not necessarily mean that the allocation is unblocked with respect to the true preferences of the participants.

In this chapter we have introduced the GSC mechanism and have established that it has some shortcomings. In the next chapter we will discuss these flaws and propose possible improvements.

8

PROBLEMS AND SOLUTION PROPOSALS

We previously presented the current mechanism being used at the GSC, described its properties and looked at the results from the pilot rotation. Several features hint at possible shortcomings of the mechanism. In this chapter we will discuss these features and try to pinpoint their cause. We will then suggest some ways of solving these shortcomings.

8.1 SIGNS OF STRUCTURAL SHORTCOMINGS IN THE GSC MECHANISM

Through conversations with the HR department and from evaluating the results of the pilot rotation, we identified the following four features.

- *No worker ranks more than five contracts,* as can be seen in Table 7.2. This is surprising, since workers were encouraged to rank more than five jobs (2017c, p.3).
- *There is a high percentage of 1:1 matches.* This is also evident from Table 7.2. Out of the 11 matched worker-job pairs, 7 are 1:1 matches.
- *One worker chose to leave the GSC on personal grounds rather than participate in the rotation exercise.*
- *There is no voluntary participation.* All workers in the pilot rotation were required to participate according to the participation criteria (2017f; 2017j).

8.2 POSSIBLE EXPLANATIONS

These features can be explained by the properties we introduced in Chapter 7.

- *No worker ranks more than five contracts.* This can be explained by the fact that the GSC mechanism is strategizable. Consider the following two cases. First, if participants have less than 5 acceptable contracts, they might want to submit the least possible amount of unacceptable contracts and only rank the minimum 5. Second, if workers have more than 5 acceptable contracts, they might truncate their submitted rankings with the intent of improving their assigned contract. For example, workers might believe that truncating their submitted ranking helps them get a more preferred contract. Overall it is difficult for workers to figure out how to best submit their preferences.
- *There is a high percentage of 1:1 matches.* This can be explained by the fact that the GSC mechanism is unstable, does not respect individual rationality with respect to the initial allocation and is strategizable. All these aspects together cause great uncertainty about how the final allocation will look like, so participants prefer to search for certainty

by talking to each other beforehand and agreeing on ranking each other as their first choices, to ensure a certain match. This evidence is similar to the phenomenon found by Ergin and Sönmez (2006) and Roth (1991) in their analysis of different priority matching mechanisms. These mechanisms were thought to be successful because they made it seem like many participants got their top choice. However, the participants got their top choices with respect to their submitted preferences and not their true ones. Similarly, in the GSC mechanism truth-telling is not incentivized, so participants strategize and by that create the impression of a well-functioning mechanism.

- *One worker chose to leave the GSC rather than participate in the rotation exercise.* There are many reasons why someone would decide to leave the GSC. It is impossible to tell whether this leave was due to the design of the mechanism or not. However, in the GSC mechanism it is possible that a worker gets assigned to a job she finds unacceptable, as we saw in Chapter 7. This might lead a worker to prefer leaving the GSC rather than risk being assigned to an unacceptable job.
- *There is no voluntary participation* because the GSC mechanism does not respect individual rationality with respect to the initial allocation. Since participants risk being assigned to a contract they like less than their initial allocation contract, it is natural that this deters them from participating, especially in a rotation that is being run for the first time.

8.3 IMPROVEMENTS

We now propose two alternative mechanisms which address different shortcomings of the GSC mechanism.[1] We discuss the trade-offs of implementing each alternative.

8.3.1 *The cumulative offer mechanism*

Our first suggestion is to replace the GSC algorithm with the COA, which was introduced in Chapter 4. In this way, we can define the *cumulative offer mechanism (COM)* as follows.

Strategy Spaces: Workers must rank a stipulated minimum number of contracts, greater than zero, and may not include the initial contract. Similarly, managers must also rank a stipulated minimum number of contracts, greater than zero, and may not include the respective initial contract.

Outcome function: The outcome of the COA is selected.

As we saw in Chapter 4, the COA will choose an allocation that is unblocked and will remove the incentive to strategize for workers. This should be highly beneficial to the entire rotation exercise since workers can openly express their preferences and do not need to waste time and effort to figure out how to best rank contracts. Given that there cannot be any blocking pairs, this suggestion increases fairness, which can possibly increase trust in the entire rotation exercise. Furthermore, the COM is easy to implement because only the algorithm of the mechanism has to be changed.[2]

1 We used the approach discussed in Chapter 6 for the construction of these mechanisms. However, since the goal of this thesis is to provide useful suggestions, we will refrain from presenting the direct mechanisms, and instead only present the corresponding and implementable indirect mechanisms.

2 Applying the COA to the pilot rotation preferences is not meaningful, since the new mechanism would have altered the entire incentive structure and submitted preferences would likely have been different.

However, two of the main problems of the GSC mechanism remain: individual rationality and individual rationality with respect to the initial allocation are still not respected. This means that, first, workers might be assigned an unacceptable contract and second, there is still no incentive to participate voluntarily. These issues stem from the fact that, independent of the used algorithm, the HR department requires all participating workers to change their job positions. As long as this requirement exists, participating workers must fear being assigned to an unacceptable contract and participation rates among volunteers will stay low. Our next suggestion will provide a possible solution to this problem by weakening this requirement.

8.3.2 *The adjusted cumulative offer mechanism*

In order to make participation attractive, workers who are thinking about joining the rotation as volunteers need the certainty of not getting a job they find worse than their current job. This is why we introduce a distinction between voluntary and mandatory workers' participation. If a worker decides to opt in voluntarily, he will be allowed to keep his initial job if he wants to. Mandatory workers do not enjoy this privilege. To ensure that a volunteer can keep his initial job, we will adjust the preferences of his respective manager as described in Chapter 5. In this way, we define the *adjusted cumulative offer mechanism (ACOM)* as follows.

Strategy space of the volunteer workers: Volunteers can rank as many contracts as they want. They must rank their initial contract last.

Strategy space of the mandatory workers: Mandatory workers must rank a stipulated minimum number of contracts and may not rank their initial contracts.

Strategy space of the managers: Managers must rank a stipulated minimum number of contracts and may not rank their initial contracts.

Outcome function: The outcome of the COA with adjusted preferences, as introduced in Chapter 5 is selected. However, only the priority ranks of the jobs that are occupied with volunteers at the beginning of the rotation exercise are adjusted. The priority rankings of the jobs occupied with mandatory workers stay the same.

To summarize the mechanism,

1. Workers' participation is divided into voluntary and mandatory. If a worker decides to opt in voluntarily, he enjoys the advantage of being allowed to keep his initial job. Mandatory participants must switch jobs.
2. Managers rank their most preferred candidates out of those who applied to their jobs.
3. The managers' submitted preferences are adjusted for the calculation of the algorithm, that is: the initial contracts of volunteers are placed at the top of the priority order of their initial job. All other contracts move one rank down.
4. Using adjusted preferences, the outcome of the ACOM is selected by running the COA with adjusted preferences.

Due to the requirement that mandatory workers must switch jobs, individual rationality is not respected for this group of workers, so we might still see workers choosing to leave the GSC. Additionally, following the same argumentation as in Chapter 5, the final outcome of the ACOM can be blocked. However, as discussed, these blocking pairs can only involve the managers of volunteers. This means that, if all volunteers decide to keep their initial job, we have the same situation as if these volunteers would not have participated in the rotation exercise to begin with. The potential blocking pairs resulting from the ACOM are thus not as disruptive as they would be in a classical matching setup.

Turning to the main advantage of the ACOM, voluntary participation is now encouraged. Workers can gain by participating in a rotation as volunteers. Therefore, workers might want to join the rotation exercise before being required to do so. This should increase the participation rate of volunteers.

8.4 COMPARISON OF DIFFERENT MECHANISMS

Recall from Chapter 1 that the pilot rotation at the GSC exhibited two main undesirable features: a low participation rate and possible signs of strategic manipulation. Both of the mechanisms we have suggested bring improvements to the currently used GSC rotation exercise, since both remove the problem of strategic manipulation from the workers. When weighing these two alternatives, one has to keep two aspects in mind. On the one hand, the ACOM encourages voluntary participation. On the other hand, the COM will be the easier alternative to implement, since only the algorithm calculations would have to be changed. Ultimately, the goals and constraints of the implementing authority will determine which alternative is more preferred.

The table below summarizes and compares the properties of the different proposed mechanisms and the GSC mechanism. In the table, *IR* stands for 'individual rationality' and *IR w.r.t. IA* stands for 'individual rationality with respect to the initial allocation'.

	Main advantage	Properties for	Strategizable	IR	IR w.r.t. IA	Unblocked
GSC Mechanism	-	*workers*	yes	no	no	no
		managers	yes	no	no	
Cumulative Offer Mechanism	*No blocking coalitions*	*workers*	no	no	no	yes
		managers	yes	no	no	
Adjusted Cumulative Offer Mechanism	*Encourages voluntary participation*	*volunteers*	no	yes	yes	no
		mandatory	no	no	no	
		managers	yes	no	no	

Table 8.1: Comparison of mechanism properties

8.4.1 *Minor improvement to the GSC mechanism*

While seeking to remove the shortcomings of the GSC mechanism requires some fundamental alterations like the ones we proposed in the previous sections, there may be a way to improve it by slightly modifying the currently used procedure.

The consideration regards the exact timing of when participants submit their rankings. Currently, workers submit a ranking of contracts before interviews are held. It is possible, however, that during the interviews a worker might gain some new information about the jobs she applied to, and might change her mind about her previously submitted ranking. For example, she might discover that she gets along particularly well with some manager, and might want to rank that job higher than she previously did. To enable workers to express their preferences when they are best informed, we suggest that they should be allowed to resubmit a new ranking of contracts after the interviews. This way of proceeding has been standard in numerous mechanisms that assign young doctors to hospitals in the US and the UK: doctors first apply to several hospitals they like, interviews are held and only after interviews both doctors and hospital submit their preference rankings to the central clearinghouse (Roth, 1984, 1991).

This concludes our solution proposals. Python scripts for all algorithms presented in this thesis can be found under: `https://github.com/pazophobie/internal-job-rotation`.

9

CONCLUSION

In this thesis we formally analyzed the job rotation exercise at the General Secretariat of the Council of the European Union. We found that the low participation rate in the rotation and the indices of strategic manipulation by participants probably originate from the shortcomings of the currently used GSC mechanism. The low participation rate stems from the fact that the currently used mechanism does not respect individual rationality with respect to the initial allocation. Since the GSC mechanism incentivizes strategizing, it is plausible that the signs of strategic manipulation are indeed evidence of participants trying to improve their outcome by submitting preferences different from their true ones.

We therefore proposed two alternative mechanisms, the cumulative offer mechanism and the adjusted cumulative offer mechanism, which target these two problems and by that improve the outcome of the rotation exercise at the GSC. The COM always produces allocations that are unblocked. The ACOM encourages voluntary participation by guaranteeing that volunteers may keep their current job in the case that a more preferred job is not found. Additionally, both mechanisms incentivize truth-telling for workers. The analysis presented in this thesis prompted a number of additional questions that would be relevant to look at in the future and would complement the understanding attained so far of the job rotation exercise at the GSC.

While our proposed model is able to capture and explain the GSC rotation exercise in its basic form, it would be interesting to extend the analysis to a dynamic game where several exercises are conducted. Furthermore, we based our model on several fundamental assumptions on the preferences of the involved participants. Especially for the managers' preferences it might be possible to find assumptions that better approximate the real situation. This would allow for deeper insights on how to construct a satisfying rotation exercise. Finally, we assumed that the mobility rates should be increased through a rotation exercise that is conducted once a year. However one could think of implementing a continuous trading platform, where workers indicate their willingness to switch jobs throughout the year. This platform could potentially be used to devise some form of job trading cycle. It would be interesting to know whether such a solution could be implemented.

Matching theory has gained significant importance in the past decade, especially due to the breadth of its applicability. However, its true potential is most evident when it is used to improve markets that affect consequential parts of our lives, like in our case, the job we go to every day. In this sense, we hope this thesis is a meaningful contribution.

BIBLIOGRAPHY

Aygün, Orhan and Tayfun Sönmez (2012). "Matching with contracts: The critical role of irrelevance of rejected contracts." In: *Working Paper, Boston College*.

Aygün, Orhan and Bertan Turhan (2017). "Large scale affirmative action in school choice: Admissions to IITs and its matching-theoretical problems." In: *American Economic Review Papers & Proceedings* 107.5, pp. 210–213.

Dimakopolous, Philipp D. and C.-Philipp Heller (2014). "Matching with waiting times: The German entry-level labour market for lawyers." In: *Working Paper, Humboldt University of Berlin*.

Ergin, Haluk and Tayfun Sönmez (2006). "Games of school choice under the Boston mechanism." In: *Journal of Public Economics* 90, pp. 215–237.

Gale, D. and L.S. Shapley (1962). "College admissions and the stability of marriage." In: *The American Mathematical Monthly* 69.1, pp. 9–15.

Hassidim, Avinatan, Assaf Romm, and Ran I. Shorrer (2017). "Redesigning the Israeli psychology master's match." In: *American Economic Review Papers & Proceedings* 107.5, pp. 205–209.

Hatfield, John W. and Fuhito Kojima (2010). "Substitutes and stability for matching with contracts." In: *Journal of Economic Theory* 145, pp. 1704–1723.

Hatfield, John W. and Paul R. Milgrom (2005). "Matching with contracts." In: *The American Economic Review* 95.4, pp. 913–935.

Hatfield, John W., Scott D. Kominers, and Alexander Westkamp (2017). "Stable and strategy-proof matching with flexible allotments." In: *American Economic Review* 5, pp. 214–219.

Hirata, Daisuke and Yusuke Kasuya (2014). "Cumulative offer process is order-independent." In: *Economics Letters* 124, pp. 37–40.

Kamada, Yuichiro and Fuhito Kojima (2012). "Stability and strategy-proofness for matching with constraints: A problem in the Japanese medical matching and its solution." In: *American Economic Review Papers & Proceedings* 102.3, pp. 366–370.

— (2015). "Efficient matching under distributional constraints: Theory and applications." In: *American Economic Review* 105, pp. 67–99.

— (2017). "Recent developments in matching with constraints." In: *American Economic Review Papers & Proceedings* 107.5, pp. 200–204.

Klaus, Bettina and Flip Klijn (2005). "Stable matchings and preferences of couples." In: *Journal of Economic Theory* 121, pp. 75–106.

Kominers, Scott D. and Tayfun Sönmez (2016). "Matching with slot-specific priorities: Theory." In: *Theoretical Economics* 11.2, pp. 683–710.

Mas-Collel, Andreu, Michael D. Whinston, and Jerry R. Green (1995). *Microeconomic theory.* New York, NY, USA: Oxford University Press.

Maskin, Eric S. (2015). "Friedrich von Hayek and mechanism design." In: *The Review of Austrian Economics* 28, pp. 247–252.

Nisan, Noam et al. (2007). *Algorithmic game theory.* New York, NY, USA: Cambridge University Press.

Pathak, Parag A. and Tayfun Sönmez (2008). "Leveling the playing field: Sincere and sophisticated players in the Boston mechanism." In: *The American Economic Review* 98.4, pp. 1636–1652.

Roth, Alvin E. (1984). "The evolution of the labor market for medical interns and residents: A case study in game theory." In: *Journal of Political Economy* 92, pp. 991–1016.

— (1991). "A natural experiment in the organization of entry-level labor markets: Regional markets for new physicians and surgeons in the United Kingdom." In: *The American Economic Review* 81.3, pp. 415–440.

— (2002). "The economist as engineer: Game theory, experimentation, and computation as tools for design economics." In: *Econometrica* 70.4, pp. 1341–1378.

— (2015). *Who gets what and why: The new economics of matchmaking and market design.* New York, NY, USA: Houghton Mifflin Harcourt Publishing Company.

Roth, Alvin E. and Elliot Peranson (1997). "The effects of the change in the NRMP matching algorithm." In: *The Journal of the American Medical Association* 278.9, pp. 729–732.

— (1999). "The redesign of the matching market for American physicians: Some engineering aspects of economic design." In: *The American Economic Review* 89.4, pp. 748–780.

Roth, Alvin E. and Marilda Oliveira Sotomayor (1990). *Two-sided matching: A study in game-theoretic modeling and analysis.* New York, NY, USA: Cambridge University Press.

Schlegel, Jan C. (2015). "Contracts versus salaries in matching: A general result." In: *Journal of Economic Theory* 159, pp. 552–573.

Sönmez, Tayfun (2013). "Bidding for army career specialties: Improving the ROTC branching mechanism." In: *Journal of Political Economy* 121, pp. 186–219.

Sönmez, Tayfun and Tobias B. Switzer (2013). "Matching with (branch-of-choice) contracts at United States military academy." In: *Econometrica* 81.2, pp. 451–488.

The General Secretariat of the Council (2017a). *A new mobility policy - A comprehensive approach to mobility.* Microsoft PowerPoint file.

— (2017b). *AD Rotation – Quick Guide for Managers.* Intranet.

— (2017c). *AD Rotation – Quick Guide for Participants.* Intranet.

— (2017d). *Decision 36/17.*

— (2017e). *GSC Internal Note - 04/12/2017.*

— (2017f). *GSC Internal Note - 13/11/2017.*

— (2017g). *GSC Internal Note - 18/01/2017.*

— (2017h). *Staff Note 46/17.*

— (2017i). *Staff Note 57/17.*

— (2017j). *Staff Note 73/17.*

— (2018a). URL: `http://www.consilium.europa.eu/en/general-secretariat/` (visited on 02/06/2018).

— (2018b). *Matching Grid - First Round.* Microsoft Excel file.

— (2018c). *Rotation Exercises at the GSC.* Intranet.

— (2018d). *Rotation for Generalist ADs - Infographic with Explanations.* Intranet.

The Lisbon Treaty - Treaty on European Union (2007).

The Lisbon Treaty - Treaty on the Functioning of the European Union (2007).

A

APPENDIX

A.1 INSTITUTIONAL BACKGROUND

The European Union (EU) is made up of seven institutions: (1) the European Council, (2) the Council of the European Union, (3) the European Parliament, (4) the European Commission, (5) the Court of Justice of the European Union, (6) the European Central Bank, and (7) the Court of Auditors (The Lisbon Treaty - Treaty on European Union, 2007, Art. 13). Within the scope of this thesis only institutions (1) and (2) are relevant.

The European Council consists of the Heads of State or Government of the Member States and meets twice every six months. Its main task is to provide the Union with the necessary impetus for its development and to define the general political directions and priorities thereof. However, it does not exercise any legislative functions (The Lisbon Treaty - Treaty on European Union, 2007, Art.15). *The Council of the European Union* consists of a representative of each Member State at the ministerial level, and meets in different configurations depending on the topic under consideration. The Council of the European Union exercises legislative and budgetary functions, carries out policy-making and coordinates functions as laid down in the Treaties (The Lisbon Treaty - Treaty on European Union, 2007, Art.16). Both of these institutions are assisted by a secretariat, namely the General Secretariat of the Council of the European Union (GSC) (The Lisbon Treaty - Treaty on the Functioning of the European Union, 2007, Art.235, 240).

The GSC is headed by its Secretary-General, who is appointed by the Council of the European Union (The Lisbon Treaty - Treaty on the Functioning of the European Union, 2007, Art.240). The Secretary-General oversees seven Directorates-General (DGs), the Legal Services and a number of other departments of the GSC. Each Directorate-General is responsible for a broad topic area and is in turn divided into several specialized units. A complete structure of the GSC can be found under: `http://www.consilium.europa.eu/media/33547/gsc-organisation-chart-en.pdf`.

The body of administrative (AD) staff of the GSC can be divided into different groups distinguished by level or qualification, defined as follows (The General Secretariat of the Council, 2017d):

AD NEWCOMER: an administrative employee whose first post at the GSC is also a first post within any of the EU institutions.

AD SPECIALIST: an administrative employee who provides essential services, where formal qualifications or specialized competences or skills are necessary to perform the assigned tasks. Only AD Specialists may occupy jobs labelled as "specialised" in the job description system.

AD GENERALIST an administrative employee occupying a post which has not been labelled as "specialized" in the job description system.

A.2 MOBILITY AT THE GSC

In order for mobility to exhibit the benefits discussed in chapter 1, the GSC stipulates that it is crucial for changes in employment position to occur between posts that are sufficiently different. Officially, mobility is defined as "a considerable change in working environment, subject matter, and functions/duties, which allows the staff member to develop new skills and new areas of knowledge. As such, intra-DG reassignment to posts with a significantly different working environment are considered mobility, while a change of tasks or reassignment to a different post within the same unit is not" (2017d, Art.1). Specifically, a *considerable change in environment* is understood as a change from one unit to another. A *considerable change in subject matter* is understood as a change in the broader subject area on which the official works. Subject matters handled within one unit are considered to belong to the same broad subject matter (2018c, p.2).

A.3 PARTICIPATION CRITERIA

The pilot rotation was carried out with administrative staff only. Two sets of criteria define which staff members shall participate in a given rotation exercise: duration criteria and size criteria.

Duration Criteria: In 2017, AD generalists who had not exercised mobility in more than 10 years were automatically included in the preliminary list of participants. AD generalists with less than 10 years but more than 2, could participate voluntarily. New AD generalists who had been in their post for more than 5 years were also automatically included in the preliminary list of participants. New AD generalists with less than 5 years but more than 2, could participate voluntarily. An exception to compulsory participation is made for AD generalists with 5 or less years left before the retirement age of 65 or 66. Over the years, the GSC aims to gradually decrease the compulsory participation threshold (2017h, Annex I, Art. 2).

Table A.1 illustrates the duration criteria of the first rotation exercise (the pilot rotation) and the duration goals for future rotations.

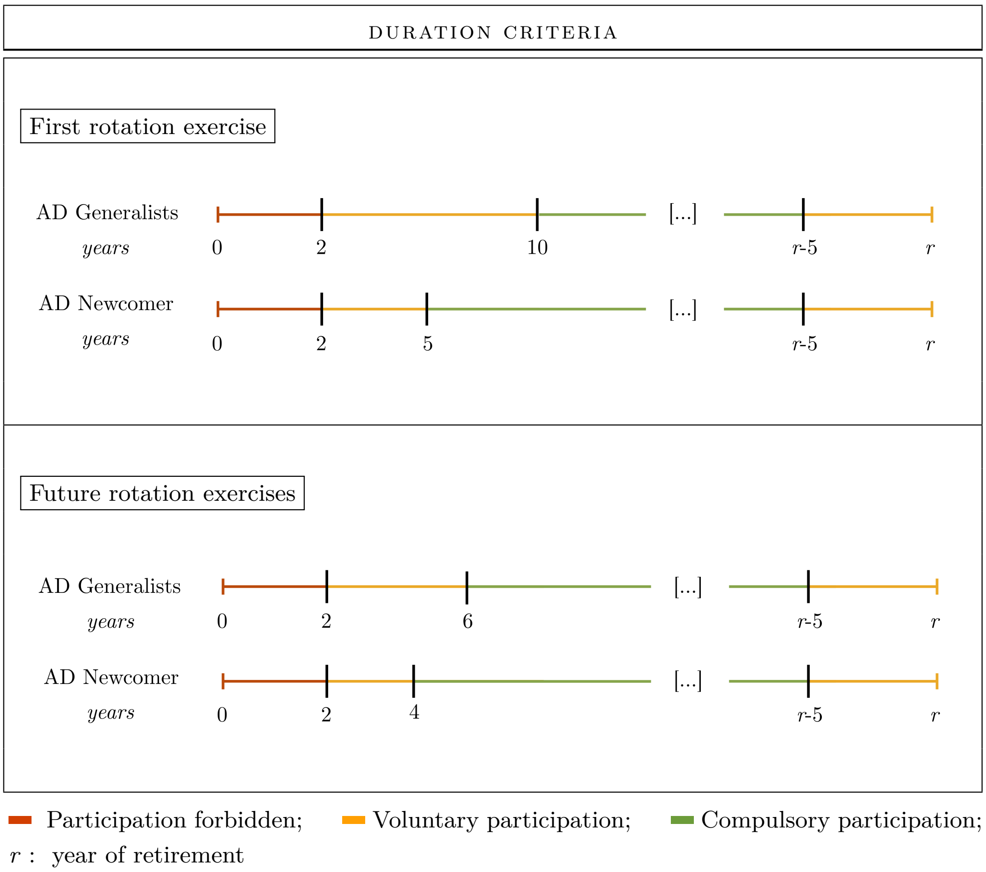

Figure A.1: Duration criteria in the GSC

Size Criteria: To ensure that the workflow of a unit is not significantly disrupted due to rotating staff, caps on participation apply according to a unit's size. If the number of total officials from one unit participating in the rotation is greater then the cap for that unit, volunteer participants will not be included in the rotation (2017b). If the number of staff for rotation is larger than the defined ceiling, officials with the highest seniority on the post shall participate in the rotation exercise (2017g, Annex II - 3.c).

SIZE CRITERIA	
Number of ADs in Unit	Max. number of ADs to participate in the rotation
≤ 4	1
$5 - 9$	2
≥ 10	3

Table A.1: Caps on participation in the rotation exercise

A.4 THE GRID

Priority	Rank on P_w	Rank on Π_j	Priority	Rank on P_w	Rank on Π_j
1	1	1	⋮	⋮	⋮
2	1	2	33	6	3
3	2	1	34	7	2
4	1	3	35	8	1
5	2	2	36	3	7
6	3	1	37	4	6
7	1	4	38	5	5
8	2	3	39	6	4
9	3	2	40	7	3
10	4	1	41	8	2
11	1	5	42	9	1
12	2	4	43	4	7
13	3	3	44	5	6
14	4	2	45	6	5
15	5	1	46	7	4
16	1	6	47	8	3
17	2	5	48	9	2
18	3	4	49	5	7
19	4	3	50	6	6
20	5	2	51	7	5
21	6	1	52	8	4
22	1	7	53	9	3
23	2	6	54	6	7
24	3	5	55	7	6
25	4	4	56	8	5
26	5	3	57	9	4
27	6	2	58	7	7
28	7	1	59	8	6
29	2	7	60	9	5
30	3	6	61	8	7
31	4	5	62	9	6
32	5	4	63	9	7
⋮	⋮	⋮			

Source: The General Secretariat of the Council, 2017i

Table A.2: The grid.