

Portfolio Optimization with Consumption and Bequest Objectives: A Monte Carlo Simulation Approach

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Abstract

This paper studies optimal portfolio allocation for high-net-worth individuals balancing lifetime consumption with bequest targets. We model this decision by combining a quadratic loss function for deviations from a targeted bequest with a CRRA utility function for consumption. Monte Carlo simulations are employed to determine wealth trajectories and then identify optimal combinations of consumption and risky asset allocation under various scenarios. We find that risk aversion has a strong effect on consumption and portfolio weights, while time preference has little impact. Higher bequest targets lead to lower consumption and higher risk-taking behavior. Similarly, symmetric penalties for deviating from the bequest target lead to higher allocation to risky assets and an increased annual consumption. Investment horizons impact the optimal portfolio distinctively, with longer horizons favoring more conservative asset allocations and consumption. Higher excess return implies a slight increase in consumption and a decrease in risky allocation. Lastly, our model is scalable across different client profiles.

Keywords: Portfolio Optimization; Bequest Motives; Optimal Consumption; Quadratic Loss Function; CRRA Utility Function; Monte Carlo Simulation; Intertemporal Choice.

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1 Introduction

Portfolio optimization has remained a dominant finance research theme over the years, in which traditional investment models such as Markowitz's Mean-Variance Optimization (1952) and Merton's Intertemporal Portfolio Choice (1969) formed the foundational basis in this domain. These models mainly focus on maximizing the mean while minimizing the variance of a portfolio over a single period. In parallel, there are numerous studies with the focus on individuals' savings behavior after retirement, examining precautionary savings and/or intentional bequest motives (Campbell and Veicera, 2003; Cocco et al., 2005; Ameriks et al., 2011). Despite these contributions, both categories of research contribute inadequately to wealth management practice, as bequests are considered as mere preferences rather than an objective that is explicitly modeled and accounted for in portfolio optimization frameworks.

Recent data and market trends indicate that global wealth management assets are valued at \$1.8 trillion in 2023, with a compound annual growth rate (CAGR) of 11.69% since 2018 and a projected CAGR of 7.03% through 2028 (Research and Markets, 2025). These insights confirm the increasing relevance of developing new portfolio optimization models that capture the dual objectives of high-net-worth individuals. Thus, this study aims to bridge the gap between wealth management academia and practical financial advisors.

This study is developed in collaboration with Lyra Financial Wealth, an independent, knowledge-based financial advisory firm that has a unique approach of cash flow-based planning and long-term asset-liability matching, supported by Monte Carlo simulations and similar tools. Their distinctive approach is important to account for the dual mandate faced by high-net-worth individuals, that is, to balance robust consumption streams while preserving wealth for inheritance purposes. However, such sophisticated approaches are not common industry practices, as other firms often remain simplistic in their methodologies, where the assessment of clients' risk preferences and product recommendations is done without thoroughly addressing long-term consumption levels or bequest motives. Moreover, very few firms globally engage in advanced asset-liability matching, which is considered a high-level approach that is typically reserved for large pension funds rather than individual clients.

1.1 Research Motivation and Contribution

The motivation for this research arises from the current identifiable gap in the literature regarding portfolio optimization for high-net-worth individuals given their dual objectives. Classical models emphasize consumption-based utility maximization (Merton, 1971; Campbell and Viceira, 2003) while they ignore the formal treatment of bequest motives, which is a key factor for wealthy households with long investment horizons. Current empirical evidence (De Nardi, 2004) points out that bequest motives and intergenerational transfers significantly influence wealth accumulation patterns, particularly among wealthy households.

This paper contributes to the literature in two ways. Firstly, it introduces a refined and practitioner-oriented framework using well-established theoretical models, such as the wealth function and the Constant Relative Risk Aversion (CRRA) utility function, while simplifying the parameters used to better reflect the real-world setting of high-net-worth individuals. In contrast to many prior studies that incorporated labor income, this approach excludes labor income, which reflects the assumption that affluent individuals have accumulated sufficient wealth. The allocation of the risky asset is held constant throughout the investment horizon, reflecting portfolio rebalancing, while consumption is modeled as a fixed percentage of initial wealth, meaning that it remains constant over the investment horizon. Likewise, a bequest is defined as a percentage of initial wealth, thus, making both consumption and bequest levels endogenous to the models. These simplifications are done in consultation with the financial advisory firm, which can enhance tractability while preserving realism, align the model more closely with current financial advisory practices. Secondly, it proposes a novel modeling approach that introduces a quadratic bequest loss function within the optimization framework. This inclusion is driven by the need to clearly address bequest motives when solving the portfolio optimization problem. The loss function is designed to be scalable with wealth because of the simplifications mentioned previously, which makes it applicable across a wide range of client profiles (see **Appendix I**). This extended model has the potential to be used as a sophisticated tool to address the dual goal of balancing sustained consumption and legacy preservation for high-net-worth individuals. Thus, our findings aim to support financial advisors in designing investment strategies to ensure a structured approach to achieving intergenerational wealth transfers.

1.2 Research Framework

The central research question that this paper addresses is: What is the optimal portfolio allocation and consumption plan that maximizes an investor's expected utility, given their predefined bequest motives, risk profile, and preferences?

We then modify preference parameters, such as risk aversion, temporal preferences, bequest levels, and preferences for bequest, to investigate their impact on investor's expected utility. Likewise, we examine how non-preference parameters such as investment horizons and excess returns affect optimal portfolio and consumption decisions. This research question, with its extensions, was developed given the limited literature addressing the dual objectives of maintaining sustainable consumption throughout the investment horizon and achieving targeted bequest outcomes among high-net-worth individuals.

We use a sequential optimization approach as the basis of our research framework which integrates three key components: a wealth evolution function, quadratic loss function for the bequest, and consumption-based (CRRA) utility function. To identify optimal strategies, we want to maximize the expected CRRA utility of consumption and minimize the expected loss of the bequest given the initial wealth, which grows according to the wealth function without labor income. The investment in the risky asset is constant over time, reflecting rebalancing of the portfolio, and consumption is fixed as a fraction of initial wealth. However, if wealth falls below the threshold of consumption, consumption adjusts dynamically as a fraction of current wealth, reflecting uncertainty. Furthermore, bequest is also reflected as a fraction of initial wealth.

This methodology and the model parameters were developed with the financial advisory firm, whose client profiles and practical experience with high-net-worth individuals revealed the need for investment models that simultaneously account for consumption preferences and bequest goals. This partnership ensures that our framework is both theoretically rigorous and grounded in the real-world needs and preferences of high-net-worth individuals.

The first step is to implement a Monte Carlo simulation that generates 10,000 paths of asset returns via Brownian motion, where we use consistent random seed across simulations to isolate parameter effects from stochastic variation. This step is essential to account for the expectation of excess returns. In the second step to solve the model we simulate wealth trajectories under various combinations of risky asset allocation and consumption rates. The terminal wealth or the actual bequest for each trajectory is evaluated through a quadratic loss function that

penalizes deviations from a predefined bequest target, with asymmetric penalties for overshooting versus undershooting, reflecting utility in the last period of the investment horizon. Then, for each parameter configuration, we identify the optimal combinations of risky asset allocation and consumption rate that minimize expected loss across all simulated paths. These optimal combinations are consequently evaluated through the CRRA utility function to determine which allocation-consumption pair maximizes the expected utility. This two-stage approach ensures that bequest objectives and consumption preferences are explicitly incorporated into the optimization process.

Finally, we develop a mathematical scaling technique that enables the application of our findings to different levels of initial wealth without requiring additional simulations, which enhances the practical use of our results for financial advisors (see **Appendix I**).

1.3 Key Findings

Our analyses highlight several key points regarding optimal portfolio allocation with respect to consumption strategies and bequest motives for high-net-worth individuals. First, temporal preference does not have a material effect on optimal portfolio choice, while the risk aversion of an investor is a critical determinant of risky share allocation and consumption level. Specifically, the optimal risky share plateaus when the risk aversion coefficients fall between 3 and 10, reaching another plateau at 15, reflecting diminishing sensitivity to risk preferences, as investors prioritize bequest goals over further reductions in risk exposure. Second, the analysis of bequest preference symmetry shows that when investors are averse to overshooting as much as to undershooting their bequest target, then as consumption increases, allocation to risky assets increases in return. This result reflects a more balanced risk-taking, which means that asymmetric penalties for deviations in bequest targets have a noticeable impact on the optimal portfolio. Third, bequest preferences substantially impact portfolio decisions, which means when investors increase their bequest target, we observe an increase in risky asset allocation, combined with a decrease in consumption. This reflects a shift toward long-term wealth preservation at the expense of current consumption. Moreover, the analyses of the investment horizon's impact on optimal portfolio uncover those investors with the shortest time horizons (i.e., 10 years) tilt their entire portfolio allocation into risky assets. This allocation declines systematically, reaching 25% when the investment period is 50 years, reflecting the profound relevance of the compounding returns. Lastly, higher excess

returns result in both higher consumption and reduced allocation to risky assets because investors can achieve their target returns with lower risk exposure. This finding suggests the creation of a positive wealth effect that alters the consumption-savings tradeoff.

1.4 Structure

The remainder of the paper is organized as follows. Section 2 covers in-depth review of the related literature. Section 3 discusses the models' assumptions and setup, as well as explaining the econometric approach to predict the optimal portfolio. Section 4 includes the empirical results of the simulations. Section 5 discusses the findings, implications, and limitations of the paper. The paper concludes in Section 6.

2 Literature Review

In this paper, we seek to investigate how optimal portfolio allocation is influenced by certain parameters, including an affluent investor's risk profile, annual consumption, and targeted bequest. To the best of our knowledge, there is no existing literature that has specially studied the optimal portfolio allocation question under the combined influence of these variables. Nevertheless, there are studies on several related topics that are highly relevant to our research question and provide a strong foundation for this paper.

2.1 Optimal Portfolio Allocation

Portfolio allocation is one of the most fundamental questions in finance, and it has been extensively studied in academic literature. This line of research traces back to the seminal work of Nobel Prize laureate Harry Markowitz (1952), who introduced mean-variance portfolio optimization in his paper *Portfolio Selection*. In his work, Markowitz (1952) argued that rational investors need to take both expected return and risk into account when constructing portfolios. One of the key implications of this paper is that optimal portfolios should be diversified in order to achieve the highest expected return for a given level of risk, forming the so-called efficient frontier, along which all investors' investment portfolios should lie regardless of their risk profiles (Markowitz, 1952).

While Markowitz (1952)'s mean-variance framework laid the foundation for modern portfolio theory, one major limitation of this framework is the single-period focus, which does not consider the dynamic nature of investment decisions over time as most investors, if not all, make investment decisions in more than one investment period. Therefore, we believe that it is essential to draw on studies that examine multi-period and continuous-time portfolio models given the long-term nature of the research question of this paper, which enables portfolio choices to be adjusted across multiple time intervals or continuously in response to ever-changing market conditions.

In multi-period models, time is broken into different intervals, allowing for sequential portfolio adjustments at fixed time periods, as well as considering transaction costs and market changes (Li et al., 2022). In another paper, Mossin (1968) demonstrated that the optimal asset allocation is independent of initial wealth, investment horizon and savings behavior under several assumptions, including constant relative risk aversion and frictionless markets. Under a similar set of assumptions, Samuelson (1969) confirmed Mossin's (1968) findings using the dynamic

stochastic programming method. However, Hakansson (1971) came to a different conclusion than the previously mentioned two scholars, stating that intertemporal preferences have a significant influence on portfolio allocation decisions.

In the field of continuous-time models, Merton (1969) suggested that the portfolio allocation remains constant for an investor with CRRA under a specific stochastic environment in his model, which aligns with Samuelson's (1969) findings in the multi-period setup. Cvitanić and Karatzas (1992) discussed how to find the optimal portfolio and consumption choice in continuous-time under certain investment constraints, such as, limits on borrowing and short-selling. Another paper written by Zhou and Li (2000) showed how to balance risk and return in a continuous-time portfolio model using linear-quadratic control methods.

2.2 Optimal Portfolio Allocation and Bequest Motives

While there may not be many studies that specifically investigate how bequest motives directly influence optimal portfolio allocation, there are several academic papers in related fields that may provide valuable insights into this relationship.

Dynan et al. (2002) argued that precautionary saving and bequest motives are inherently intertwined during uncertain times, and that the existence of a precautionary saving motive can significantly diminish the additional impact of bequest motives on the wealth accumulation for most households. This implies that bequest motives have a limited effect on optimal portfolio allocation that in turn affects a household's total wealth. However, De Nardi (2004) shows that bequest motives play a central role particularly among high-net-worth households, in shaping wealth accumulation and saving behaviors. Furthermore, Ameriks et al. (2011) demonstrated that low consumption among retired individuals can be explained by strong bequest motives, indicating the importance of bequest motive on consumption which is a critical parameter in portfolio optimization questions.

While Cocco et al.'s (2005) paper focused primarily on the life cycle aspect of the portfolio optimization problem, they also considered bequest motives in their framework, showing the impact of bequest considerations on portfolio allocation. They noted in the paper that investors with a targeted bequest motive may have relatively more savings than those without bequest motives, which in turn influences optimal portfolio allocation as it is related to wealth accumulation that can be affected by bequest intentions (Cocco et al., 2005). Nevertheless, Cocco

et al. (2005) pointed out that the significance of bequest is highly debated within the academic circle. Scholars such as Hurd (1989), Cagetti (2002) and Dynan et al. (2002) have shown that bequest motives have minimal or no impact on household saving behavior, and thus, limited influence on portfolio decisions (Cocco et al., 2005). Cagetti (2002) demonstrated that the overall saving behavior changed very little even for investors with strong bequest motives, because cautious individuals already maintain large wealth buffers.

Another implication of Cocco et al.'s (2005) paper is that the bequest motive influences saving behavior differently across the life cycle. For instance, young people, who generally save less due to their tendency for immediate consumption, would increase their savings when a bequest motive is present (Cocco et al., 2005). Meanwhile, for middle-aged investors, Cocco et al. (2005) found that the bequest motive has less influence over them, as they tend to accumulate significant savings at this stage of life regardless of inheritance intentions. For retired individuals, on the other hand, the bequest motive has a greater impact on portfolio allocation, compared to previous life stages, due to a decline in precautionary saving needs and an enhanced awareness of mortality risk (Cocco et al., 2005).

Furthermore, Cocco et al. (2005) argued that young investors tend to allocate a smaller share of their portfolio to equities compared to middle-aged investors, with this effect being particularly pronounced among those with bequest motives, highlighting the influence of inheritance intentions on portfolio decisions.

2.3 Optimal Portfolio Allocation and Consumption

Given the importance of consumption in shaping individual wealth dynamics and financial planning, a substantial body of literature has examined portfolio optimization in conjunction with consumption decisions (e.g., see Mossin, 1968; Samuelson, 1969; Merton, 1969; Cvitanić and Karatzas 1992; Cocco et al., 2005; Dempster, 2016).

Hakansson (1971) highlighted that optimal consumption and portfolio choices are highly sensitive to both individual preferences and the nature of investment opportunities over time. Moreover, Merton (1971) demonstrated that under the assumption of CRRA, the optimal proportion of wealth invested in risky assets remains constant over time, which can be achieved through continuous portfolio rebalancing. However, his analysis omitted transaction costs, assuming frictionless markets in which continuous portfolio rebalancing can occur without

incurring any trading costs (Merton, 1971). Mei and Nogales (2018) improved Merton's framework by including transaction costs in their model.

Jeon and Park (2020) investigated optimal retirement timing and portfolio-consumption decisions for an agent who avoids any decline in consumption. Two key findings of this paper are that the agent's implied risk aversion remains constant over time, and the proportion of wealth allocated to risky assets increases as wealth grows (Jeon and Park, 2020).

Similar to our study, Chacko and Viceira (2015) examined how long-term investors make optimal decisions regarding consumption and portfolio allocation, assuming access to a risk-free asset, such as bonds, with a constant return, and a risky asset, such as stocks, with a stable expected return but fluctuating uncertainty, quantified by time-varying precision (the inverse of volatility).

Chacko and Viceira (2015) acknowledged that existing literature suggests long-term investors tend to allocate more to risky assets than single-period investors, partly to hedge their future consumption against changes in investment opportunities. However, their empirical results indicate that the actual impact of intertemporal hedging demand is small because changes in volatility are not large or persistent enough to significantly adjust investment choices (Chacko and Viceira, 2015). In contrast, studies by Brandt (1999), Campbell and Viceira (1999, 2001, 2002), and others have found much larger hedging demands, driven by time-varying risk premia or real interest rates (Chacko and Viceira, 2015).

3 Methodology and Theoretical Framework

To address the research question, we solve the model by maximizing the expected CRRA utility while minimizing the expected loss with respect to the bequest. The model begins with an initial wealth level and a bequest target set as a fixed percentage of initial wealth, as provided by the financial advisory firm, and assumes no labor income, reflecting the reality of high-net-worth individuals. Portfolio returns are modeled with a fixed proportion allocated to risky assets, maintained consistently throughout the investment horizon. Consumption is set as a fixed percentage of the initial wealth, reflecting a preference for consistency in cash flow. Although consumption remains constant under normal conditions, it adjusts to a fraction of current wealth when wealth falls below threshold level, thereby introducing uncertainty into the consumption path. This deviation is penalized under CRRA preferences because the utility function is highly sensitive to fluctuations in consumption. The framework's assumptions are simplified compared with those in the existing literature, but they are more aligned with practitioner's practical approach.

3.1 Model Specification and Theoretical Framework

3.1.1 *Wealth Function*

In the traditional formulation of the wealth function, an investor's wealth over a specific period is given by their initial wealth and gross return on their investment portfolio (Campbell and Viceira, 2002). This foundational approach is frequently used in dynamic portfolio choice models when wealth changes only due to investment returns, without taking consumption into account. Other studies which have built upon this framework incorporate a range of real-world frictions as well as additional components. For example, some literature extends the wealth function to include labor income and examine life-cycle behavior (Cocco et al., 2005). These extensions allow for a more nuanced understanding of wealth accumulation and consumption choices over the life cycle.

This traditional framework serves as the foundation for our paper, which we extend on by explicitly taking into consideration annual consumption patterns. We assume that consumption occurs at the beginning of each period, and that wealth at the end of each period is calculated as wealth at the beginning of the period minus consumption, multiplied by the gross return on a portfolio comprised of risk-free and risky assets. This adjustment allows for a more realistic representation of how consumption decisions impact portfolio dynamics over time and aligns our framework with the broader literature on portfolio optimization with respect to consumption. Our

model also excludes labor income for high-net-worth individuals to better capture the wealth dynamics of this investor group. This approach is supported by the financial advisory firm, who confirmed that labor income plays a trivial role in their clients' total financial wealth.

Wealth Function:

$$W_{t+1} = R_{t+1}^P (W_t - C_t)$$

Where R_{t+1}^P denotes the gross return on the investor's portfolio at period $t+1$ from period t

$$R_{t+1}^P = (1 - w) R_f + w r_{t+1}^e$$

Subject to the following constraints:

$$C_t = \min (C^*, a W_t)$$

$$W_{t+1} \geq 0$$

In this function, (W_{t+1}) represents the total wealth of an investor's portfolio in period $t+1$, which depends on the wealth from the previous period (W_t) , the risk-free rate (R_f) , the average excess return on risky assets (r_{t+1}^e) , the portfolio allocation to risky assets (w) , and the yearly withdrawal for consumption (C_t) .

The first constraint ensures that consumption never falls below zero, since it is defined as such that the investor consumes the lesser of a fixed amount which is a fraction of initial wealth $(C^* = xW_0)$ or a predetermined percentage (a) of current wealth at time t (i.e., aW_t). Specifically, when total wealth exceeds (C^*) level, the investor consistently withdraws (C^*) ; however, if wealth falls below this threshold, consumption is adjusted to fraction (a) of the available current wealth (W_t) . This constraint allows wealth to recover over time, as the investor consumes a fraction of the available wealth rather than the entire wealth available at time t , thereby minimizing the probability of wealth approaching zero, which is a necessary condition for the following analysis using the CRRA function. The second constraint ensures that wealth never becomes negative at any point in time, which is in line with the model presented in Cagetti's (2003) paper.

3.1.2 Quadratic Loss Function for Bequest

Cocco et al.'s (2005) paper assumes the same utility function for an investor's consumption and bequest motive. While other study modeled preferences for consumption and bequests using isoelastic utility functions with a relative risk aversion coefficient of 3, reflecting high risk aversion, while the annual time preference rate and the interest rate are assumed to be 3%, indicating a balanced view between present and future consumption (Dynan et al., 2002).

In contrast to the aforementioned approaches, our model adopts a distinct quadratic loss function for bequests rather than applying the same utility specification used for consumption. This distinction allows us to quantify the expected utility loss associated with the investor's bequest objective, as individuals may value leaving a bequest differently from how they value personal consumption during their lifetime.

Quadratic Loss Function for Bequest:

$$L(W_T, X, s, u) = -s \{[\min(W_T - X, 0)]^2 + u [\max(W_T - X, 0)]^2\}$$

This quadratic loss function for bequest seeks to examine the degree of dissatisfaction an investor experiences if their actual bequest fails to meet the target level. Specifically, the investor experiences a loss of utility if the actual amount of the bequest deviates from their predetermined target. The unique feature of the loss function is its asymmetric treatment of overshooting versus undershooting the bequest target, allowing for the hypothesis that investors dislike undershooting the target more than overshooting it, which is the standard assumption in this paper. This assumption is supported by client experiences in financial advisory, where high-net-worth individuals typically have a stronger aversion to falling short rather than exceeding their bequest target.

In this function, (W_T) is the actual wealth investor has at the end of their investment period, that is the realized bequest, while (X) represents the targeted bequest level, which is a predefined fraction of the initial wealth (zW_0) . Parameter (s) captures the penalty for deviating from the bequest target, whereas (u) specifically, indicates the penalty for overshooting. This dual-parameter framework allows for different assessments of undershooting versus overshooting the bequest target.

3.1.3 CRRA

In the existing literature, the CRRA utility function is a commonly used model for solving portfolio optimization and optimal consumption questions (Merton, 1969; Cocco et al., 2005; Mei and Nogales, 2018).

Building on the foundational work of Merton (1969), we adopt the optimal strategy for investors with CRRA preferences, which is to maximize the expected utility of consumption for the investor. Similar to the approach taken by Mei and Nogales (2018), our model is designed to address a dynamic portfolio optimization problem within a finite-horizon setting, where the investor aims to maximize the expected CRRA utility derived from yearly consumption.

However, our paper differs methodologically: we first simulate the wealth function and the loss function, then we identify the optimal combinations of risky asset allocation and yearly consumption. For these obtained optimal combinations, we evaluate the CRRA utility function to identify the optimal portfolio that yields the highest CRRA utility. Thus, through this stepwise approach, our study focuses on determining both the optimal consumption levels and the optimal allocation to risky assets given a set of predefined parameters.

CRRA Utility Function:

$$U(C_t) = \sum_{t=1}^T \beta^t \frac{C_t^{(1-\gamma)}}{(1-\gamma)}$$

The setup of the CRRA function used in this paper follows the one presented in Cagetti's (2001) paper, where (β^t) is the discount factor and (γ) is the relative risk aversion.

3.2 Research Design and Empirical Implementation

3.2.1 Monte Carlo Simulation Framework

To address the paper's main research question on optimal portfolio allocation and consumption planning, we implement Monte Carlo simulation, which is a comprehensive framework that captures the dual objectives of achieving the bequest target and maximizing the utility of consumption. Monte Carlo simulations represent a well-established tool for modeling financial

uncertainty, which allows us to generate thousands of potential wealth trajectories under various market conditions. This approach provides a probabilistic perspective that deterministic models cannot offer, which enables investors and wealth advisors to make informed judgments about the likelihood of achieving financial objectives.

We utilize a sequential optimization approach that integrates three key components that were developed in collaboration with the financial advisory firm: a wealth evolution function that tracks portfolio dynamics over time, a quadratic loss function that quantifies deviations from bequest targets, and a consumption-based CRRA utility function. Our collaboration with the financial advisory firm ensured that our framework reflects both theoretical rigor and the practical preferences exhibited by high-net-worth clients.

The simulation framework operates in different stages. In the first stage, we generate 10,000 paths of excess returns following a log-normal distribution via Brownian motion, which is a continuous-time stochastic process that is used in financial modeling. Each path represents a possible realization of market performance over the investor's time horizon. We save these simulated return paths and reuse them across all scenarios, employing the same random number seed to ensure that variations in outcomes can be attributed only to changes in tested parameters rather than to differences in the underlying return sequences.

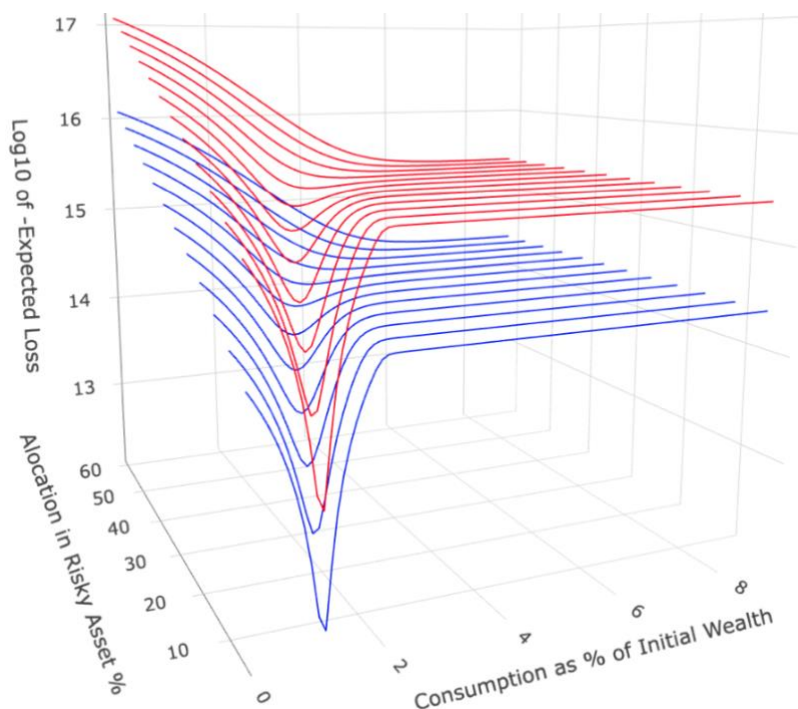
In the second stage, we explore different combinations of risky asset allocations and consumption rates in order to test various investment strategies. Specifically, the portfolio allocation to risky assets is set as a sequence with annual rebalancing to maintain the chosen allocation. Similarly, the annual withdrawal for consumption is set as a sequence, that is a percentage of the initial wealth, which is important for the consistency of our model. For each possible combination of risky asset allocation and consumption rate, we apply our wealth function across all 10,000 pre-generated return paths, effectively running 10,000 distinct simulations per combination per year, to trace the evolution of wealth over the investor's time horizon. This approach is comprehensive as it produces a robust distribution of outcomes for each investment strategy, which ensures that our conclusions are not driven by anomalous return sequences.

In the third stage, we take the terminal wealth (wealth at time T) from each simulated trajectory and assess it through the proposed quadratic loss function, which penalizes deviations from the predefined bequest target. By taking the average of the loss value across all 10,000 simulations for each combination of consumption and allocation in risky assets, we calculate the

expected loss associated with each strategy. This allows us to identify the optimal combinations that minimize the expected bequest loss across all simulated paths.

To support the practical application of our approach, we show the scalability of the results, allowing the findings to be applied consistently across clients with different initial wealth levels, without the need to re-run simulations for every profile with new initial wealth. This is made possible by the structure of the quadratic loss function, which scales proportionally with wealth due to the model simplifications discussed earlier (see **Figure 1** and **Appendix I**).

Figure 1: Log Scaled 3D Plot for Expected Loss



Note: This figure shows consumption as % of initial wealth (x-axis, 0.1% to 9% by 0.1%) and risky asset allocation (y-axis 5% to 60% by 5%). The expected loss (logarithmically scaled z-axis) is computed as the average loss from the quadratic loss function across 10,000 Monte Carlo simulations. The loss function penalizes deviations of actual terminal wealth at time (T) from a predefined bequest target (50% of initial wealth), with undershooting penalized twice as heavily as overshooting. The same investment horizon (50 years) and identical return paths are used across the simulations. Client 1 (blue) has an initial wealth of 32M; Client 2 (red) has 100M. Given that the red and blue lines follow the same trajectories, the graph illustrates how loss values are scaled proportionally by a constant factor across different initial wealth levels, confirming the model's scalability. See Appendix I for further details.

Lastly, we take these optimal combinations and subsequently evaluate them through the CRRA utility function to determine which allocation-consumption pair maximizes the expected utility that is derived from yearly consumption. This approach ensures that both bequest objectives and consumption preferences are explicitly incorporated into the optimization process.

3.2.2 Comparative Statistics

In this section, we define the different scenarios used to evaluate the impact of varying investor preferences and market conditions on optimal consumption and portfolio allocation. The base case is seen as the benchmark, while in all alternative modifications we change one key parameter at a time to isolate their effect.

In the base case scenario, we try to solve the portfolio's optimal risky share and optimal annual consumption with predefined initial wealth, bequest target, investment horizon, excess returns of the risky assets, volatility of the risky assets, beta, gamma, and a penalty structure for deviations from the bequest goal.

To perform comparative statics analysis, we group the parameters into preference and non-preference categories and vary one parameter at a time to observe its impact on the optimal portfolio allocation, while keeping all other base-case parameters constant. An outline of these parameters and the corresponding analyses is presented below:

- Preference parameters:
 - Beta and gamma (Risk Aversion and Temporal Preference Analysis)
 - Bequest goal (Bequest Target Analysis)
 - Penalty parameters for deviations from bequest goal – s and u (Bequest Preference Symmetry Analysis)
- Non-preference parameters:
 - Investment horizon (Investment Horizon Analysis)
 - Excess return (Excess Return Analysis)

Base Scenario Analysis

In the base case we apply a predefined set of assumptions for the parameter values for the three functions defined previously.

Wealth Function Parameters:

We assume an initial wealth (W_0) of 32 million, which is in line with the financial advisory firm's high-net-worth client profiles. Then, we assume that the investor targets a bequest of 50% of initial wealth, i.e., 16 million, with an investment horizon of 50 years. The risk-free rate is assumed to be 0%, accounting for transaction costs and taxes, which also reflects the current low-interest-rate environment and is in line with the financial advisory firm's data. The expected excess return on risky assets is set at 5%, with a standard deviation of 15%. We set the portfolio allocation to risky assets as a sequence from 5% to 100% increasing by 5% with annual rebalancing to maintain the allocation. While the annual withdrawal for consumption is a sequence ranging from 0.1% to 15% of initial wealth, increasing by 0.1% of initial wealth. Lastly, the fraction of current wealth that investors would consume (a) is set to 50%, which is the case if current wealth (W_t) falls below the consumption threshold (C^*).

Quadratic Loss Function Parameters:

We assume that (s) which governs the degree of aversion to deviations from the target bequest is set to 1. The asymmetry parameter (u) is set to 0.5 which indicates that the investor is twice as averse to undershooting the target as to overshooting it. These parameters mirror the reality and reflect client preferences observed in the financial advisory. i.e., the situation where the investor experiences significantly greater dissatisfaction when the bequest falls short of the target, as opposed to when it exceeds the target. However, overshooting the predefined bequest target is still penalized.

CRRA Function Parameters:

We assume that the risk aversion coefficient (γ) is set to 2. Since this parameter determines the investor's level of risk aversion, 2 is deemed to be appropriate reflecting a moderate risk aversion, where higher value of (γ) indicates greater risk aversion. Moreover, (β) represents the discount factor, is set to 0.98. This parameter reflects the investor's preference for present consumption over future consumption, with values closer to 1 indicating a preference for future consumption.

Risk Aversion and Temporal Preference Analysis

In order to assess the impact of investor preferences of consumption on the optimal portfolio, we change the CRRA utility function parameters, i.e., risk aversion (γ) and temporal preference (β), while keeping all other parameters fixed at the base case level. Specifically, (γ) ranges from 1.5 to 5 increasing by 0.5, then it ranges from 5 to 15 increasing by 5, then it increases from 15 to 30 by 15, while (β) ranges from 0.95 to 0.99 increasing by 0.01.

Bequest Target Analysis

To evaluate how stronger bequest motives influence consumption and asset allocation, we increase the target bequest to 100% of initial wealth that is 32 million, with all other parameters remaining constant from the base case.

Bequest Preference Symmetry Analysis

In this scenario, we explore the effect of symmetric dissatisfaction with respect to bequest deviations, where the investor is equally averse to both overshooting and undershooting the target. Therefore, we set ($u=s=1$) which implies a symmetric penalty structure in the quadratic loss function, while keeping all other parameters unchanged from the base scenario.

Investment Horizon Analysis

To examine how the change in investment horizon affects the optimal portfolio, we vary the investment period (T) from 10 to 50 years in 10-year increments, while keeping all other parameters the same as the base case.

Excess Return Analysis

In this case, to isolate the effect of higher expected returns on the optimal portfolio's risky asset allocation and consumption decisions, we increase the excess return from 5% to 7%, while keeping all other parameters unchanged from the base case.

4 Results

4.1 Base Scenario

Table 1: Base Case, Optimal Results

Weight in risky assets	Consumption (% of initial wealth)	Average loss values	Average CRRA values
0.05	384,000 (1.2%)	-1.68849E+12	-8.11346E-05
0.1	416,000 (1.3%)	-7.10946E+12	-7.48935E-05
0.15	480,000 (1.5%)	-1.81736E+13	-6.49077E-05
0.2	544,000 (1.7%)	-3.84107E+13	-5.72719E-05
0.25*	608,000* (1.9%)	-7.13087E+13	-5.16134E-05
0.3	672,000 (2.1%)	-1.18846E+14	-6.71383E-05
0.35	800,000 (2.5%)	-1.75958E+14	-0.003128091
0.4	992,000 (3.1%)	-2.30718E+14	-0.348359602
0.45	1,440,000 (4.5%)	-2.54562E+14	-112.8599942
0.5	2,176,000 (6.8%)	-2.55968E+14	-9199.29396
0.55	2,624,000 (8.2%)	-2.55974E+14	-41092.00137
0.6	2,848,000 (8.9%)	-2.55974E+14	-71701.65339
0.65	3,072,000 (9.6%)	-2.55975E+14	-114044.1214
0.7	3,296,000 (10.3%)	-2.55975E+14	-168956.6098
0.75	3,520,000 (11%)	-2.55975E+14	-237040.9411
0.8	3,744,000 (11.7%)	-2.55974E+14	-317435.7905
0.85	3,968,000 (12.4%)	-2.55979E+14	-409673.649
0.9	4,160,000 (13%)	-2.55969E+14	-497751.3369
0.95	4,352,000 (13.6%)	-2.55953E+14	-594260.7915
1	4,576,000 (14.3%)	-2.55928E+14	-717471.3595

Note: “” denotes the optimal value for weight allocation to risky assets and annual consumption. The first column shows the weight allocation to risky assets in an investment portfolio for a high-net-worth investor, with values ranging from 0 to 1, assuming there is a borrowing constraint. The second column shows the annual consumption amount, expressed as a percentage of the initial wealth, which remains constant over the entire investment period. The third column illustrates the loss in utility when the actual bequest amount at the end of*

the investment period deviates from the bequest target. The loss values depend on the penalty structure, which reflects the investor's preference for undershooting versus overshooting the target. In the base case, the investor dislikes undershooting twice as much as overshooting. The fourth column shows the average CRRA value for each combination of consumption and weight allocation to risky assets. The average CRRA values demonstrate the utility the investor gains from a given consumption level. Since bequest motives are present, an investor must strike a balance between annual consumption and the probability of meeting the bequest goal, which depends on the weight of risky assets. This weight is in turn partly influenced by the investor's preferences for investment risk and consumption, namely risk aversion and a preference for present over future consumption. Given that the investor wants to maximize consumption and minimize loss associated with deviations from the bequest target, the model generates 20 combinations of weight allocation and consumption level that provide the lowest average loss values and highest average CRRA values among all simulation paths. Based on this, the combination with the highest average CRRA value will be the optimal portfolio choice for the investor.

In the base case, the investor has an initial wealth of 32 million and a targeted bequest of 50% of the initial wealth, i.e., 16 million. As mentioned in the methodology section, an investor seeks to maximize her expected CRRA utility, meaning the case that generates the highest CRRA value is considered optimal. Assuming the investment horizon is 50 years, with a gamma of 2, beta of 0.98, the optimal risky share and consumption are 25% and 1.9% of initial wealth (i.e., 608,000 in absolute terms), respectively as shown in Table 1.

4.2 Risk Aversion and Temporal Preference Analysis

In this scenario, we aimed to investigate the impact of investor preferences on optimal portfolio allocation and annual consumption by varying two preference parameters: gamma and beta. The gamma value increases by 0.5 from 1.5 to 5, then from 5 to 15 by 5, and jumps to 30, reflecting a shift from a more risk-taking to a strongly risk-averse profile. Similarly, the beta value changes by 0.01 from 0.95 to 0.99, representing the investor's level of patience. As beta decreases from 0.99 to 0.95, the investor discounts future wealth more heavily, implying less patience. The resulting optimal portfolio allocations i.e., proportion of portfolio invested in risky assets and consumption rates for different combinations of beta and gamma are presented in Table 2.

When the coefficient of relative risk aversion (gamma) is reduced to 1.5 from the base case, the optimal allocation consists of an annual consumption level of 2.1% of initial wealth and a portfolio weight of 30% in risky assets. This combination yields the highest expected CRRA value,

indicating that a less risk-averse investor prefers both higher consumption and greater exposure to risky assets. This optimal configuration remains consistent across a range of discount factors (beta) from 0.95 to 0.99.

Conversely, for an investor with a higher degree of risk aversion, represented by a gamma of 2.5, the optimal consumption and portfolio allocation remain unchanged from the base case (gamma = 2). Specifically, the optimal annual consumption is 1.9% of initial wealth, and the risky asset share is 25%. This outcome holds for all beta values within the same range (0.95 to 0.99), suggesting that modest increases in risk aversion do not necessarily alter the optimal investment-consumption trade-off under these parameters.

Table 2: Different Risk and Temporal Preferences with Optimal Results

Gamma	Beta	Optimal Weight in Risky Asset	Optimal Consumption (%)
1.5	0.95-0.99	0.30	672,000 (2.1%)
2	0.95-0.99	0.25	608,000 (1.9%)
2.5	0.95-0.99	0.25	608,000 (1.9%)
3	0.95-0.99	0.20	544,000 (1.7%)
3.5	0.95-0.99	0.20	544,000 (1.7%)
4	0.95-0.99	0.20	544,000 (1.7%)
4.5	0.95-0.99	0.20	544,000 (1.7%)
5	0.95-0.99	0.20	544,000 (1.7%)
10	0.95-0.99	0.20	544,000 (1.7%)
15	0.95-0.99	0.15	480,000 (1.5%)
30	0.95-0.99	0.15	480,000 (1.5%)

Note: The first column shows different gamma values, reflecting an investor's risk aversion. The higher the gamma, the more risk averse the investor is. The Beta column presents different beta values ranging from 0.95 to 0.99, with values closer to 1 indicating a stronger preference for future consumption. The last two columns show the optimal weight of risky assets and the optimal annual consumption (as a percentage of initial wealth) for different combinations of gamma and beta values. The optimal choice is determined in the same way as described in the Methodology and Theoretical Framework section, that is, based on both the average loss values and the average CRRA values. Only the optimal results are shown here, see Appendix II for detailed information.

As the investor becomes significantly more risk averse, with gamma values ranging from 3 to 10, the optimal allocation shifts. The share of wealth invested in risky assets declines to 20%, while the optimal annual consumption decreases to 1.7% of initial wealth (i.e., 544,000). Importantly, this allocation remains stable across the entire beta spectrum from 0.95 to 0.99, indicating that changes in temporal preferences (patience) have little influence on the optimal policy within this risk aversion range. Only when the gamma takes on extreme values, such as 15 and 30, the optimal weight of risky assets is reduced to 15% and the consumption lowers to 1.5% of initial wealth.

In summary, the investor's level of risk aversion (gamma) has a stronger impact on their optimal consumption and investment decisions than their level of patience (beta). As gamma increases, the ideal portion of wealth invested in risky assets goes down, leveling off at 20% for gamma values between 3 and 10. At the same time, optimal annual consumption also decreases, matching the lower exposure to risk, which shows a consistent relationship between risk tolerance and spending behavior. As risk aversion increases extremely high, to 15 and 30, the optimal combination changed to an allocation of 15% to risky assets and an annual consumption of 1.5% of initial wealth.

4.3 Bequest Target Analysis

Increasing the bequest amount to be 100% of initial wealth i.e., equivalent to 32 million, while keeping all other parameters constant, results in an optimal portfolio allocation of 35% in risky assets and a consumption of 544,000 corresponding to 1.7% of initial wealth. The increase in bequest target to 100% of initial wealth suggests that an investor would decrease their consumption while increasing their exposure to risky assets. This behavior reflects a shift in focus toward allocating more wealth for future inheritance, while taking on additional risk to achieve this goal, as illustrated in Table 3.

Table 3: Higher Bequest, Optimal Results

Weight in risky assets	Consumption (% of initial wealth)	Average loss values	Average CRRA values
0,05	64,000 (0,2%)	-2,523E+12	-0,000486808
0,1	128,000 (0,4%)	-1,15662E+13	-0,000243404
0,15	224,000 (0,7%)	-2,83911E+13	-0,000139088
0,2	288,000 (0,9%)	-5,79086E+13	-0,000108179
0,25	384,000 (1,2%)	-1,05098E+14	-8,11346E-05
0,3	448,000 (1,4%)	-1,7516E+14	-6,95439E-05
0,35*	544,000 (1,7%)*	-2,80205E+14	-5,74269E-05
0,4	608,000 (1,9%)	-4,26498E+14	-6,33467E-05
0,45	736,000 (2,3%)	-6,15159E+14	-0,002489921
0,5	896,000 (2,8%)	-8,28198E+14	-0,177744633
0,55	1,280,000 (4,0%)	-9,90819E+14	-44,07279159
0,6	2,432,000 (7,6%)	-1,02344E+15	-23605,41286
0,65	2,720,000 (8,5%)	-1,02377E+15	-52828,87827
0,7	3,264,000 (10,2%)	-1,0239E+15	-160071,497
0,75	3,488,000 (10,9%)	-1,0239E+15	-226254,9948
0,8	3,712,000 (11,6%)	-1,0239E+15	-304758,9558
0,85	3,936,000 (12,3%)	-1,02389E+15	-395378,5952
0,9	4,128,000 (12,9%)	-1,02387E+15	-482046,3654
0,95	4,320,000 (13,5%)	-1,0238E+15	-577512,2932
1	4,512,000 (14,1%)	-1,02364E+15	-681348,0085

Note: “” denotes the optimal value for weight allocation to risky assets and annual consumption. This table illustrates the optimal combination of weight allocation to risky assets and annual consumption when the targeted bequest increases from 50% to 100% of initial wealth. Find the note for Table 1 for the descriptive statistics of each column.*

4.4 Bequest Preference Symmetry Analysis

Table 4: Symmetrical Penalty Preference for Bequest, Optimal Results

Weight in risky assets	Consumption (% of initial wealth)	Average loss values	Average CRRA values
0,05	384,000 (1.2%)	-2,12935E+12	-0,056603798
0,1	448,000 (1.4%)	-9,98914E+12	-0,052404934
0,15	512,000 (1.6%)	-2,65766E+13	-0,049020327
0,2	576,000 (1.8%)	-5,62537E+13	-0,046217801
0,25	640,000 (2.0%)	-1,03584E+14	-0,043911179
0,3*	768000 (2.4%)*	-1,64998E+14	-0,042978069
0,35	928,000 (2.9%)	-2,25063E+14	-0,082205047
0,4	1,248,000 (3.9%)	-2,53691E+14	-1,085925767
0,45	2,144,000 (3.7%)	-2,55975E+14	-46,4382877
0,5	2,400,000 (7.5%)	-2,55975E+14	-77,79236138
0,55	2,624,000 (8.2%)	-2,55974E+14	-110,8676969
0,6	2,848,000 (8.9%)	-2,55974E+14	-148,356849
0,65	3,072,000 (9.6%)	-2,55975E+14	-189,0666045
0,7	3,296,000 (10.3%)	-2,55975E+14	-231,9450521
0,75	3,520,000 (11.0%)	-2,55975E+14	-276,2049572
0,8	3,744,000 (11.7%)	-2,55974E+14	-320,757289
0,85	3,968,000 (12,4%)	-2,55979E+14	-365,0657599
0,9	4,160,000 (13.0%)	-2,5597E+14	-401,2665726
0,95	4,352,000 (13,6%)	-2,55965E+14	-436,8333135
1	4,576,000 (14,3%)	-2,55929E+14	-478,9598735

Note: “” denotes the optimal value for weight allocation to risky assets and annual consumption. This table illustrates the optimal combination of weight allocation to risky assets and annual consumption when the investor has a symmetric penalty structure for overshooting and undershooting the predefined bequest target. Find the note for Table 1 for the descriptive statistics of each column.*

In the base case scenario, it was assumed that undershooting the bequest target is more penalized than overshooting (i.e., $u = 0.5 < s = 1$). However, in this scenario, equal penalties are applied to

both undershooting and overshooting the target (i.e., $s = u = 1$). As seen in Table 4, the optimal allocation to risky assets is 30% with a consumption level of 768,000, which corresponds to 2.4% of initial wealth. This suggests that the investor optimally increases both their risk exposure and consumption when they are indifferent between undershooting and overshooting the bequest target, i.e., when the asymmetry in penalty is removed.

4.5 Investment Horizon Analysis

In order to understand the significance of investment horizons on portfolio allocation problems, we employed a sensitivity analysis to assess the potential effects of different time horizons on the optimal combination of risky assets and annual consumption compared to the base case. In other words, we changed only one variable, the investment period, from 50 years to 10, 20, 30, and 40 years, and obtained the following results.

When the investment horizon is relatively short, specifically 10 years, the optimal portfolio allocation consists of a 100% investment in risky assets, accompanied by an annual consumption level of 9.4% of initial wealth (Table 5). As the investment horizon extends to 20 years, the optimal share of wealth allocated to risky assets decreases to 55%, while the optimal annual consumption level adjusts to 4.5% of initial wealth (Table 6). With a further increase in the investment period to 30 years, the allocation to risky assets declines to 40%, and the corresponding optimal consumption falls to 3.1% of initial wealth (Table 7). For an investment horizon of 40 years, the optimal risky asset share continues its downward trend to 30%, with optimal annual consumption reaching 2.3% of initial wealth (Table 8).

Overall, the findings indicate a clear inverse relationship between the length of the investment horizon and both the optimal share of risky assets and the level of annual consumption. As the horizon lengthens, investors tend to reduce their exposure to risky assets and adopt a more conservative consumption strategy. The resulting optimal portfolios (i.e., risky asset allocations and consumption rates) for different investment horizons are presented in Table 9. This trend reflects a longer-term perspective on wealth preservation and risk management over time.

Table 5: Investment period, T=10 years, Optimal Results

Weight in risky assets	Consumption (% of initial wealth)	Average loss values	Average CRRA values
1*	3,008,000 (9.4%)*	-1.53861E+14	-3.32789E-06
0.95	2,848,000 (8.9%)	-1.38296E+14	-3.35496E-06
0.9	2,720,000 (8.5%)	-1.23042E+14	-3.41871E-06
0.85	2,592,000 (8.1%)	-1.08275E+14	-3.52448E-06
0.8	2,496,000 (7.8%)	-9.42277E+13	-3.62738E-06
0.75	2,400,000 (7.5%)	-8.10644E+13	-3.75242E-06
0.7	2,336,000 (7.3%)	-6.89042E+13	-3.84607E-06
0.65	2,272,000 (7.1%)	-5.78652E+13	-3.94909E-06
0.6	2,208,000 (6.9%)	-4.79419E+13	-4.06099E-06
0.55	2,144,000 (6.7%)	-3.91213E+13	-4.18113E-06
0.5	2,080,000 (6.5%)	-3.13879E+13	-4.30944E-06
0.45	2,016,000 (6.3%)	-2.4697E+13	-4.44616E-06
0.4	1,984,000 (6.2%)	-1.89397E+13	-4.51786E-06
0.35	1,920,000 (6.0%)	-1.40871E+13	-4.66845E-06
0.3	1,888,000 (5.9%)	-1.00699E+13	-4.74758E-06
0.25	1,824,000 (5.7%)	-6.79218E+12	-4.91416E-06
0.2	1,792,000 (5.6%)	-4.24235E+12	-5.00192E-06
0.15	1,728,000 (5.4%)	-2.31664E+12	-5.18717E-06
0.1	1,696,000 (5.3%)	-1.00819E+12	-5.28504E-06
0.05	1,632,000 (5.1%)	-2.52317E+11	-5.4923E-06

Note: “” denotes the optimal value for weight allocation to risky assets and annual consumption. This table illustrates the optimal combination of weight allocation to risky assets and annual consumption when the investment horizon is 10 years. Find the note for Table 1 for the descriptive statistics of each column.*

Table 6: Investment period, T=20 years, Optimal Results

Weight in risky assets	Consumption (% of initial wealth)	Average loss values	Average CRRA values
1	5,472,000 (17.1%)	-2.55756E+14	-0.00497975
0.95	5,088,000 (15.9%)	-2.55696E+14	-0.003947012
0.9	4,576,000 (14.3%)	-2.55559E+14	-0.002705823
0.85	3,840,000 (12.0%)	-2.54862E+14	-0.001305956
0.8	2,752,000 (8.6%)	-2.50504E+14	-0.000226486
0.75	2,208,000 (6.9%)	-2.34113E+14	-5.51088E-05
0.7	1,920,000 (6.0%)	-2.08938E+14	-2.26867E-05
0.65	1,728,000 (5.4%)	-1.79492E+14	-1.40843E-05
0.6	1,568,000 (4.9%)	-1.49066E+14	-1.17363E-05
0.55*	1,440,000 (4.5%)*	-1.20302E+14	-1.16612E-05
0.5	1,376,000 (4.3%)	-9.42379E+13	-1.19582E-05
0.45	1,280,000 (4.0%)	-7.18516E+13	-1.27443E-05
0.4	1,216,000 (3.8%)	-5.31003E+13	-1.3397E-05
0.35	1,152,000 (3.6%)	-3.79925E+13	-1.41384E-05
0.3	1,120,000 (3.5%)	-2.60087E+13	-1.45422E-05
0.25	1,056,000 (3.3%)	-1.68557E+13	-1.54235E-05
0.2	992,000 (3.1%)	-1.01315E+13	-1.64186E-05
0.15	960,000 (3.0%)	-5.35621E+12	-1.69658E-05
0.1	896,000 (2.8%)	-2.22159E+12	-1.81777E-05
0.05	864,000 (2.7%)	-5.93812E+11	-1.88509E-05

Note: “” denotes the optimal value for weight allocation to risky assets and annual consumption. This table illustrates the optimal combination of weight allocation to risky assets and annual consumption for an investment horizon of 20 years. Find the note for Table 1 for the descriptive statistics of each column.*

Table 7: Investment period, T=30 years, Optimal Results

Weight in risky assets	Consumption (% of initial wealth)	Average loss values	Average CRRA values
1	5,696,000 (17.8%)	-2.55974E+14	-3.602808814
0.95	5,440,000 (17.0%)	-2.55974E+14	-3.147923612
0.9	5,184,000 (16.2%)	-2.55975E+14	-2.713346884
0.85	4,896,000 (15.3%)	-2.55974E+14	-2.247773383
0.8	4,640,000 (14.5%)	-2.55974E+14	-1.863013529
0.75	4,384,000 (13.7%)	-2.55974E+14	-1.50435697
0.7	4,096,000 (12.8%)	-2.55974E+14	-1.139699398
0.65	3,808,000 (11.9%)	-2.55974E+14	-0.821907127
0.6	1,760,000 (5.5%)	-2.48171E+14	-0.003560388
0.55	1,440,000 (4.5%)	-2.20527E+14	-0.000388131
0.5	1,216,000 (3.8%)	-1.82448E+14	-5.3598E-05
0.45	1,088,000 (3.4%)	-1.4159E+14	-2.522E-05
0.4*	992,000 (3.1%)*	-1.04212E+14	-2.30442E-05
0.35	928,000 (2.9%)	-7.29129E+13	-2.40704E-05
0.3	864,000 (2.7%)	-4.84154E+13	-2.57802E-05
0.25	800,000 (2.5%)	-3.02869E+13	-2.78391E-05
0.2	736,000 (2.3%)	-1.75704E+13	-3.02599E-05
0.15	704,000 (2.2%)	-9.06095E+12	-3.16353E-05
0.1	640,000 (2.0%)	-3.60096E+12	-3.47989E-05
0.05	576,000 (1.8%)	-8.62063E+11	-3.86654E-05

Note: “” denotes the optimal value for weight allocation to risky assets and annual consumption. This table illustrates the optimal combination of weight allocation to risky assets and annual consumption for an investment horizon of 30 years. Find the note for Table 1 for the descriptive statistics of each column.*

Table 8: Investment period, T=40 years, Optimal Results

Weight in risky assets	Consumption (% of initial wealth)	Average loss values	Average CRRA values
1	5,024,000 (15.7%)	-2.55957E+14	-1588.636888
0.95	4,768,000 (14.9%)	-2.55951E+14	-1328.600567
0.9	4,544,000 (14.2%)	-2.55967E+14	-1117.177303
0.85	4,288,000 (13.4%)	-2.55973E+14	-894.8493222
0.8	4,064,000 (12.7%)	-2.55973E+14	-719.8995681
0.75	3,808,000 (11.9%)	-2.55974E+14	-540.7960431
0.7	3,552,000 (11.1%)	-2.55975E+14	-387.8285205
0.65	3,328,000 (10.4%)	-2.55974E+14	-276.4056084
0.6	2,976,000 (9.3%)	-2.55963E+14	-143.8772085
0.55	2,752,000 (8.6%)	-2.55954E+14	-86.20283827
0.5	1,504,000 (4.7%)	-2.51735E+14	-0.337795847
0.45	1,120,000 (3.5%)	-2.22278E+14	-0.005707741
0.4	9,280,00 (2.9%)	-1.71677E+14	-0.000200598
0.35	8,320,00 (2.6%)	-1.21584E+14	-4.35253E-05
0.3*	736,000 (2.3%)*	-7.98172E+13	-3.71868E-05
0.25	672,000 (2.1%)	-4.84241E+13	-4.0422E-05
0.2	608,000 (1.9%)	-2.70499E+13	-4.46722E-05
0.15	576,000 (1.8%)	-1.34832E+13	-4.7154E-05
0.1	512,000 (1.6%)	-5.15887E+12	-5.30482E-05
0.05	448,000 (1.4%)	-1.14535E+12	-6.06265E-05

Note: “” denotes the optimal value for weight allocation to risky assets and annual consumption. This table illustrates the optimal combination of weight allocation to risky assets and annual consumption for an investment horizon of 40 years. Find the note for Table 1 for the descriptive statistics of each column.*

Table 9: Different Investment Periods with Optimal Results

Investment Period	Optimal Weight in Risky Asset	Optimal Consumption (%)
10	1.00	3,008,000 (9.4%)
20	0.55	1,440,000 (4.5%)
30	0.40	992,000 (3.1%)
40	0.30	736,000 (2.3%)
50	0.25	608,000 (1.9%)

Note: This table summarizes the optimal combination of weight allocation to risky assets and annual consumption for different investment periods, ranging from 10-50 years.

4.6 Excess Return Analysis

Increasing the excess return to 7%, i.e., by two percentage points from the base case of 5% while keeping all other parameters constant, we notice that the optimal combination of portfolio weight and consumption shifts to 20% and 640,000, respectively, corresponding to 2% of initial wealth. This outcome, summarized in Table 10, suggests that when the portfolio's expected return increases, the investor would increase their consumption while decreasing their exposure to risky assets as the expected return of the portfolio is supposed to increase.

Table 10: Higher Excess Return, Optimal Results

Weight in risky assets	Consumption (% of initial wealth)	Average loss values	Average CRRA values
0,05	384,000 (1,2%)	-1,93766E+12	-8,11346E-05
0,1	480,000 (1,5%)	-8,09973E+12	-6,49077E-05
0,15	480,000 (1,7%)	-2,35994E+13	-5,72715E-05
0,2*	640,000 (2,0%)*	-5,24441E+13	-4,87025E-05
0,25	736,000 (2,3%)	-1,01835E+14	-4,87762E-05
0,3	864,000 (2,7%)	-1,68271E+14	-0,000915715
0,35	1,120,000 (3,5%)	-2,31858E+14	-0,339405354
0,4	1,600,000 (5,0%)	-2,55234E+14	-111,404615
0,45	2,272,000 (7,1%)	-2,55964E+14	-5647,522272
0,5	2,720,000 (8,5%)	-2,55974E+14	-24583,58568
0,55	2,976,000 (9,3%)	-2,55975E+14	-43835,41138
0,6	3,232,000 (10,1%)	-2,55976E+14	-69876,00021
0,65	3,488,000 (10,9%)	-2,55975E+14	-102377,0239
0,7	3,744,000 (11,7%)	-2,55971E+14	-140553,1296
0,75	3,968,000 (12,4%)	-2,55947E+14	-175779,0431
0,8	4,224,000 (13,2%)	-2,55933E+14	-220390,672
0,85	4,544,000 (14,2%)	-2,55959E+14	-283828,4057
0,9	4,896,000 (15,3%)	-2,55972E+14	-358603,6526
0,95	5,216,000 (16,3%)	-2,55975E+14	-425859,5788
1	5,504,000 (17,2%)	-2,5598E+14	-482341,9222

Note: “” denotes the optimal value for weight allocation to risky assets and annual consumption. This table illustrates the optimal combination of weight allocation to risky assets and annual consumption when the excess return increases to 7% instead of 5% in the base case. Find the note for Table I for the descriptive statistics of each column.*

5 Discussion

5.1 Discussion of Findings

5.1.1 Risk Aversion and Temporal Preference Analysis

Our study finds that risk aversion (γ) has a more noticeable effect on optimal portfolio in terms of risky share allocation and consumption decisions than temporal preferences (β). When the coefficient of relative risk aversion increases from 1.5 to 10, we notice that the optimal portfolio allocation of risky assets decreases from 30% to 20%, with consumption levels similarly declining from 2.1% to 1.7% of initial wealth. This inverse relationship between risk aversion and risky asset allocation is in line with the mean-variance trade-off in Markowitz's (1952) portfolio-selection framework, where more risk-averse investors choose lower-variance portfolios. Furthermore, it is in line with Tobin's (1958) theorem, which shows that the fraction invested in the risky tangency portfolio is inversely proportional to the investor's risk-aversion coefficient. This confirms that rational investors' relative risk aversion would affect their portfolio exposure to risky assets.

These results are also consistent with Merton's (1969, 1971) findings that under CRRA utility, higher risk aversion leads to lower risky asset allocation. Another explanation for the decrease in consumption as risk aversion increases and the allocation to risky assets decreases is the presence of a bequest motive. By cutting back on consumption, the investor increases the likelihood of meeting the bequest goal, since total wealth tends to be lower with a more conservative portfolio.

However, our study reveals that the optimal allocation reaches a plateau at 20% for (γ) values between 3 and 10, suggesting diminishing effects of increased risk aversion beyond certain thresholds. We have two explanations for this plateau. First, beyond a certain (γ), further risk aversion no longer reduces the risky share materially since investors prioritize meeting the bequest goal. As shown with gamma values of 15 and 30, risk aversion must reach extreme levels before it meaningfully lowers the optimal risky share and consumption again, reaching another plateau. Second, computational constraints limited us from using intervals smaller than 5%, such as 1%, so we were unable to capture more granular changes in portfolio weights and consumption levels that may occur with increasing risk aversion, assuming those changes are relatively small. Another possible reason why consumption levels reached a plateau is that investors prefer to maintain their consumption level, as further reductions would lead to larger declines in CRRA utility (average CRRA values). That is, investors find these utility losses more undesirable than the risk of bearish

scenarios in which risky assets underperform, and they are forced to reduce consumption (i.e., consuming aW_t instead of xW_0). Nevertheless, these patterns offer nuanced insights beyond what Merton's model predicted, because we incorporate bequest targets with consumption utility.

With regard to temporal preferences, our model did not find any significant effect on portfolio and consumption optimization, which differs from the conclusion of Hakansson's (1971) paper. One possible explanation could be that our model has a different setup, such as the incorporation of bequest motives and a different methodology for optimization. However, our results revealed that different beta values led to different CRRA values (see **Appendix II**), indicating that temporal preferences can influence an investor's utility, although not significantly enough to alter the optimal combination of risky share and consumption in our model. On the other hand, it is possible that a more noticeable effect of temporal preferences on this combination could be observed if we were not subject to the limitation of computational power and could narrow the intervals for both the optimal weight of risky assets and optimal consumption, for example, allowing the weight differences to be 0.01 instead of 0.05.

5.1.2 Bequest Target Analysis

When the bequest target increased from 50% to 100% of initial wealth, the optimal allocation to risky assets increased from 25% to 35%, while consumption decreased from 1.9% to 1.7% of initial wealth. This trade-off shows that stronger bequest motives lead investors to reduce their current consumption while increasing their risk exposure to achieve higher terminal wealth.

Our findings connect to Ameriks et al. (2011), who used strategic surveys to quantify bequest motives and found substantial heterogeneity in how individuals value bequests relative to late-life consumption. Our results suggest that bequest targets do influence portfolio decisions, particularly when these targets are explicitly modeled as penalties for deviation. This finding contrasts with certain perspectives in the literature, such as Dynan et al. (2002), who argued that bequest motives have minimal incremental impact on wealth accumulation beyond precautionary savings. However, our finding is consistent with Cocco et al.'s (2005) finding who shows that modeling bequests as explicit targets with penalties has an effect on portfolio behavior, especially in later stages of the life cycle. However, it is worth noticing that our study focuses on high-net-worth individuals and does not explicitly distinguish between different stages of the life cycle in the modeling framework.

5.1.3 Bequest Preference Symmetry Analysis

The results when comparing between asymmetric penalties ($s=1, u=0.5$) and symmetric penalties ($s=u=1$) for undershooting versus overshooting the bequest target shows important behavioral insights. Under symmetric penalties, the optimal portfolio risky share increases from 25% to 30% and consumption rises from 1.9% to 2.4% of initial wealth. This indicates that when investors are equally concerned about exceeding and falling short of bequest goals, they implement more aggressive investment and consumption strategies. In contrast, when undershooting the bequest target is penalized more heavily, investors adopt more conservative strategies to safeguard against the risk of failing to meet their goal. These results highlight how the relative weight investors place on different types of bequest deviations can significantly shape both asset allocation and consumption decisions, even when overall preferences and constraints remain otherwise unchanged.

5.1.4 Investment Horizon Analysis

Our results show a clear inverse relationship between investment horizons and optimal portfolio allocation. For a shorter investment period of 10 years, the optimal share in risky assets is 100% with high consumption equivalent to 9.4% of initial wealth. As the investment period prolongs to 50 years as in our base case, the optimal share in risky assets declines to 25% with a significantly lower annual consumption equivalent to 1.9% of initial wealth. This finding may seem counterintuitive, as longer investment horizons are typically associated with higher risk tolerance and, therefore, greater allocation to risky assets compared to shorter horizons. However, the compounding effect of returns may help explain this phenomenon. Returns generated at the end of each year are reinvested in the portfolio, aside from the portion withdrawn for annual consumption. This creates a compounding effect that becomes stronger over longer investment horizons. As a result, the optimal allocation to risky assets can be lower for longer-term portfolios while still achieving the bequest goal. The observed decrease in consumption with longer horizons reflects the need to spread consumption over a longer period, during which the investor consumes for a longer period, but holds a lower risky share.

5.1.5 *Excess Return Analysis*

Our results show that increasing the excess return from 5% to 7% results in a decline in allocation of risky assets from 25% to 20%, while consumption faced a slight increase from 1.9% to 2.0% of initial wealth. This decrease in risky asset allocation despite higher expected returns could be explained by the wealth effect, where higher expected returns let investors increase the present value of future wealth, allowing them to meet their bequest targets with lower risk exposure, which enables them to increase consumption as well. This suggests that higher expected returns allow investors to reduce risk exposure while maintaining or slightly increasing their consumption levels.

Our findings are in line with Zhou and Li's (2000) work on balancing risk and return in continuous-time portfolio models, where optimal allocation depends not just on risk-return tradeoffs but also on overall wealth targets. The results also show that when bequest targets are explicitly incorporated, the relationship between expected returns and optimal risky allocation is not necessarily monotonic. This finding contrasts with the classical mean-variance framework of Markowitz (1952), where higher expected returns typically lead to higher allocations to risky assets.

5.2 **Key Implications**

One key implication of this paper is that risk aversion has a more significant effect on optimal portfolio allocation and consumption decisions than temporal preferences. As risk aversion increases, investors allocate less to risky assets and reduce consumption, which is consistent with classical theories. Yet, further increases in risk aversion do not significantly change the optimal allocation to risky assets beyond a certain threshold ($\gamma = 3$ to 10) as investors prioritize meeting bequest targets over additional risk reduction. Only when risk aversion becomes extremely high does the optimal allocation to risky assets and consumption shift to a lower plateau. This finding contributes to existing literature by incorporating bequest motives, revealing a plateau effect not predicted by traditional frameworks. However, temporal preferences (β) show minimal impact on portfolio and consumption choices in this model, contrasting with some prior literature. For practitioners in financial wealth advisory, this implication can serve as supporting evidence to demonstrate to highly risk-averse clients that a certain allocation to risky assets is essential to pursuing their bequest objectives.

As the model setup prioritizes bequest targets, investors should increase their investments in risky assets while cutting back on current spending when those targets become more ambitious relative to initial wealth. This reflects the impact of a strong bequest motive on consumption and the trade-off mechanism that can be quantified by financial advisors.

Additionally, the results reveal that the penalty structure for missing or exceeding bequest targets has a significant influence on behavior. In other words, symmetric penalties encourage riskier asset allocation and more aggressive consumption, while asymmetric penalties, which place a greater penalty on undershooting, lead to more conservative investing and spending structure. As these findings illustrate how penalty preferences related to bequest motives can reshape portfolio allocation and consumption behavior, it is essential for wealth managers to take these penalty preferences into account when constructing portfolios and financial plans for their clients.

Apart from that, the results suggest that investment horizons play a crucial role in portfolio optimization problems, demonstrating that shorter horizons lead to higher risky asset allocations and greater consumption. We believe this outcome may be attributed to the compounding effect of returns. However, future research is encouraged to further validate this implication.

Furthermore, our results regarding excess returns during varying market conditions imply that higher expected returns allow investors to both reduce risky asset exposure while slightly increasing consumption levels. This is explained by the fact that investors capitalize on greater expected wealth to meet long-term bequest goals with less risk, rather than increasing expected wealth. For wealth managers, this means that portfolio allocation should not automatically increase equity exposure during periods of higher expected returns if bequest objectives exist.

Lastly, the quadratic loss function framework developed in this paper offers an innovative way to capture bequest motives and measure its impact on portfolio allocation. By introducing explicit penalties for deviating from a targeted bequest level, the model captures the practical reality where investors have specific bequest goals, rather than aiming to maximize the utility of final wealth. The framework exhibits consistent scaling properties, meaning that optimal consumption and portfolio choices remain proportional even as an investor's wealth changes (see **Appendix I**). This makes the model especially useful for financial advisors, as the optimization results can be applied broadly across clients with different wealth levels.

5.3 Limitations

While our paper contributes to the existing literature with certain interesting implications, several limitations should be acknowledged. First, the model assumes constant risk aversion over the investment horizon, but risk preferences may change due to market conditions or as investors approach their terminal wealth or bequest target in the real-world scenario. If this assumption is relaxed by applying a dynamic risk aversion framework in our model, the results may be different and more nuanced. Second, our model is restricted to only a risk-free and a single risky asset, which does not reflect multiple asset classes and different factor portfolio choices in the real world. Because of time constraints and data limitations, we were not able to produce a more refined analysis by incorporating other asset classes or segmenting the equity portfolio based on different factors. Third, our model assumes continuous portfolio rebalancing, which may not reflect real-world constraints, such as transaction costs. Moreover, transaction costs and taxation were simplified in this model as we account for them by setting the risk-free to zero, which may be insufficient given the complexity of the tax environment for high-net-worth investors. Therefore, a more detailed treatment of tax effects would enhance the model's practical applicability, given different tax jurisdictions. Fourth, while bequest motives are essential to the findings, isolating their impact by comparing models with and without bequest constraints could strengthen causal inferences. Lastly, computational constraints restricted the granularity of portfolio weights and consumption level, which suggests that future research with enhanced computational resources could expand on our results. Despite these limitations, our study makes a significant contribution to the literature by demonstrating how optimal portfolio allocation is mutually influenced by risk aversion, bequest targets, attitudes toward deviating from these targets, investment horizon and excess returns. This means that as we integrate goal-based investing principles with traditional portfolio theory, our findings offer both theoretical insights and practical guidance for long-term wealth management.

6 Conclusion

This paper investigates the optimal portfolio allocation and consumption strategy that maximizes expected utility while minimizing the expected loss for high-net-worth individuals with explicit bequest motives. We use a sequential optimization framework by combining a wealth evolution function with a quadratic loss function for bequest and CRRA utility function to identify optimal strategies through Monte Carlo simulations, supported by insights from the financial advisory firm.

The research yields several key implications: risk aversion has a more significant effect on optimal portfolio allocation and consumption decisions than temporal preferences, revealing a plateau phenomenon. Higher bequest targets relative to initial wealth require increased investment in risky assets and a reduction in the consumption level. Additionally, the bequest penalties' structure impacts the optimal portfolio allocation, where symmetric penalties allow for risk-taking, while asymmetric ones lead to conservative portfolio allocation. Shorter investment horizons lead to increased allocations to risky assets and an increase in consumption, likely reflecting the compounding effect of returns. Higher expected returns slightly increase the consumption level and allow for reduced risk exposure. Lastly, our quadratic loss function framework provides a scalable approach that allows the findings to be applied consistently across clients with different initial wealth levels.

The findings of this paper reveal several opportunities for future exploration given the limitation of our current framework. Risk aversion could vary throughout the investment horizon to reflect changing risk preferences due to market conditions or proximity to terminal wealth targets. Moreover, to better reflect real-world multiple asset classes and factor portfolio choices, one could expand the model beyond a risk-free and single risky asset. Also, accounting for transaction costs and more complex tax treatments in the model would enhance its applicability across different tax jurisdictions. Moreover, a comparative analysis of models with and without bequest constraints would strengthen causal conclusions regarding the impact of bequest motives. Lastly, enhanced computational power may enable greater granularity in portfolio weights and consumption levels, which would in turn allow for more detailed results.

Each of these limitations offers potential for future research to further bridge the gap between theoretical models and the requirements of high-net-worth individuals faced by financial advisory services, thereby improving portfolio optimization techniques to better match clients' needs.

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Appendix I

Proof of Model Scalability: Quadratic Loss Function for Bequest

Our empirical analysis confirms a powerful scaling relationship across wealth levels. When initial wealth is scaled by a factor (k), the resulting loss function scales proportionally by (k^2). This relationship holds constant across different client profiles when using identical consumption rates and risky asset allocations.

Loss Function:

$$L(W_T, zW_0, s, u) = -s \{[\min(W_T - zW_0, 0)]^2 + u [\max(W_T - zW_0, 0)]^2\}$$

Where:

- W_T is the terminal wealth
- zW_0 is the bequest target defined as a fraction of initial wealth
- s and u are penalty parameters for deviating from the bequest target and overshooting it

Wealth Function:

$$W_{t+1} = R_{t+1}^P (W_t - C_t)$$

Where:

- W_t is the wealth at time t
- W_{t+1} is the wealth at time $t+1$
- R_{t+1}^P is the gross return on the portfolio
 - Where $R_{t+1}^P = (1 - w) R_f + w r_{t+1}^e$
 - w is the weight in risky assets
 - R_f is the risk-free rate
 - r_{t+1}^e is the excess return on risky assets
- Consumption is C_t which is the minimum between a predefined fraction of initial wealth (C^*) or a percentage of current wealth, i.e., $\min(C^*, a W_t)$

Experimental Setup

We illustrate this scaling property with two client profiles that share identical assumptions except for initial wealth:

- Client 1: $W_{01} = 32$
- Client 2: $W_{02} = 100$

Other parameters are kept identical, but scale with W_0 , which is needed in order to isolate the effect of scaling wealth in the loss outcome:

- Time horizon: $T=50$
- Bequest 50% of W_0
- Excess return: 5% with 15% standard deviation
- Share of risky assets: 5% to 100% increasing by 5%
- Consumption is % of W_0 :
 - 0.1% to 9% increasing by 0.1% when risky share is less than 65%
 - 9% to 15% increasing by 0.1% when risky share is larger or equal to 65%.
- Loss function parameters: $s=1$ and $u=0.5$

Methodology

1. Gross Return Paths: We set seed and generate paths of gross returns for the investment horizon using Monte Carlo simulation to be consistent across simulations.
2. Simulation Runs: For each combination of risky weight and consumption rate, we generate 10,000 Monte Carlo trials for both clients.
3. Loss Calculation: For each simulation path, we calculate the loss according to the loss function based on the realized terminal wealth for that specific path of gross returns.
4. Averaging: We obtain the expected loss $E(L_1)$ and $E(L_2)$ by averaging across the 10,000 simulations for each client and each parameter combination of risky share and consumption.
5. Then we compute the average loss across all 10,000 simulations for each client and each combination of risky share and consumption
6. Ratio Analysis: For each combination, we calculate the ratio of Client 2's average loss to Client 1's average loss: $[E(L_2)/E(L_1)]$

Results

For all tested combinations of risky share and consumption rates (see **Table A**), we observe that:

$$\frac{E(L_2)}{E(L_1)} = 9.765625$$

For example:

Client	Risky Share	Consumption	Consumption % of W_0	Average Loss
2	5%	100,000	0.1% of 100M	-1698656821881840
1	5%	32,000	0.1% of 32M	-173942458560700
Ratio				9.765625

This value is precisely equals to:

$$\left(\frac{W_{02}}{W_{01}}\right)^2 = \left(\frac{100}{32}\right)^2 = (3.125)^2 = 9.765625$$

Conclusion

From the empirical analysis and the results obtained, we confirm that when initial wealth is scaled by a factor k (where $k = W_{02}/W_{01}$), the resulting loss function scales by (k^2) :

$$\frac{E(L_2)}{E(L_1)} = \left(\frac{W_{02}}{W_{01}}\right)^2$$

This relationship can be expressed as follows, using the same gross return path, the same weight in risky assets and the same consumption level, with W_{01} serving as the reference case:

$$E[L(W_{02}, C^*, w)] = k^2 E[L(W_{01}, C^*, w)]$$

This implies that if the optimal portfolio (consumption rate and risky-share allocation) with respect to the loss value for a reference client is computed, then they can be applied to any other client whose wealth level is different by a constant factor. This means that if we multiply the client's initial wealth and all wealth-dependent parameters by a factor (k) , then the resulting expected loss is multiplied by (k^2) , where $k > 0$. Once we simulate the model for a “reference” client, we can immediately calculate the results for any other client whose wealth profile differs only by a constant scaling factor, without rerunning the full simulation. Therefore, the scale invariance enhances both the beauty of the model and its practical applicability for financial planning across diverse client portfolios across a wide range of portfolio sizes.

Table A: Comparison of Expected Losses for Two Clients with Different Initial Wealth

Weight	Consumption as % of Initial Wealth	Average Loss Client 2	Average Loss Client 1	Loss Ratio
0,05	0,1%	-1,699E+15	-1,739E+14	9,765625
0,05	0,2%	-1,403E+15	-1,437E+14	9,765625
0,05	0,3%	-1,136E+15	-1,163E+14	9,765625
0,05	0,4%	-8,967E+14	-9,182E+13	9,765625
0,05	0,5%	-6,863E+14	-7,027E+13	9,765625
0,05	0,6%	-5,043E+14	-5,164E+13	9,765625
0,05	0,7%	-3,508E+14	-3,593E+13	9,765625
0,05	0,8%	-2,258E+14	-2,312E+13	9,765625
0,05	0,9%	-1,293E+14	-1,324E+13	9,765625
0,05	1,0%	-6,120E+13	-6,267E+12	9,765625
0,05	1,1%	-2,214E+13	-2,267E+12	9,765625
0,05	1,2%	-1,649E+13	-1,688E+12	9,765625
0,05	1,3%	-5,497E+13	-5,629E+12	9,765625
0,05	1,4%	-1,469E+14	-1,504E+13	9,765625
0,05	1,5%	-2,954E+14	-3,025E+13	9,765625
0,05	1,6%	-5,008E+14	-5,128E+13	9,765625
0,05	1,7%	-7,631E+14	-7,814E+13	9,765625
0,05	1,8%	-1,082E+15	-1,108E+14	9,765625
0,05	1,9%	-1,458E+15	-1,493E+14	9,765625
0,05	2,0%	-1,885E+15	-1,930E+14	9,765625
0,05	2,1%	-2,264E+15	-2,318E+14	9,765625
0,05	2,2%	-2,442E+15	-2,501E+14	9,765625
0,05	2,3%	-2,487E+15	-2,547E+14	9,765625
0,05	2,4%	-2,497E+15	-2,557E+14	9,765625
0,05	2,5%	-2,499E+15	-2,559E+14	9,765625
0,05	2,6%	-2,500E+15	-2,560E+14	9,765625
0,05	2,7%	-2,500E+15	-2,560E+14	9,765625
0,05	2,8%	-2,500E+15	-2,560E+14	9,765625
0,05	2,9%	-2,500E+15	-2,560E+14	9,765625
0,05	3,0%	-2,500E+15	-2,560E+14	9,765625
0,05	3,1%	-2,500E+15	-2,560E+14	9,765625
0,05	3,2%	-2,500E+15	-2,560E+14	9,765625
0,05	3,3%	-2,500E+15	-2,560E+14	9,765625
0,05	3,4%	-2,500E+15	-2,560E+14	9,765625
0,05	3,5%	-2,500E+15	-2,560E+14	9,765625
0,05	3,6%	-2,500E+15	-2,560E+14	9,765625

0,05	3,7%	-2,500E+15	-2,560E+14	9,765625
0,05	3,8%	-2,500E+15	-2,560E+14	9,765625
0,05	3,9%	-2,500E+15	-2,560E+14	9,765625
0,05	4,0%	-2,500E+15	-2,560E+14	9,765625
0,05	4,1%	-2,500E+15	-2,560E+14	9,765625
0,05	4,2%	-2,500E+15	-2,560E+14	9,765625
0,05	4,3%	-2,500E+15	-2,560E+14	9,765625
0,05	4,4%	-2,500E+15	-2,560E+14	9,765625
0,05	4,5%	-2,500E+15	-2,560E+14	9,765625
0,05	4,6%	-2,500E+15	-2,560E+14	9,765625
0,05	4,7%	-2,500E+15	-2,560E+14	9,765625
0,05	4,8%	-2,500E+15	-2,560E+14	9,765625
0,05	4,9%	-2,500E+15	-2,560E+14	9,765625
0,05	5,0%	-2,500E+15	-2,560E+14	9,765625
0,05	5,1%	-2,500E+15	-2,560E+14	9,765625
0,05	5,2%	-2,500E+15	-2,560E+14	9,765625
0,05	5,3%	-2,500E+15	-2,560E+14	9,765625
0,05	5,4%	-2,500E+15	-2,560E+14	9,765625
0,05	5,5%	-2,500E+15	-2,560E+14	9,765625
0,05	5,6%	-2,500E+15	-2,560E+14	9,765625
0,05	5,7%	-2,500E+15	-2,560E+14	9,765625
0,05	5,8%	-2,500E+15	-2,560E+14	9,765625
0,05	5,9%	-2,500E+15	-2,560E+14	9,765625
0,05	6,0%	-2,500E+15	-2,560E+14	9,765625
0,05	6,1%	-2,500E+15	-2,560E+14	9,765625
0,05	6,2%	-2,500E+15	-2,560E+14	9,765625
0,05	6,3%	-2,500E+15	-2,560E+14	9,765625
0,05	6,4%	-2,500E+15	-2,560E+14	9,765625
0,05	6,5%	-2,500E+15	-2,560E+14	9,765625
0,05	6,6%	-2,500E+15	-2,560E+14	9,765625
0,05	6,7%	-2,500E+15	-2,560E+14	9,765625
0,05	6,8%	-2,500E+15	-2,560E+14	9,765625
0,05	6,9%	-2,500E+15	-2,560E+14	9,765625
0,05	7,0%	-2,500E+15	-2,560E+14	9,765625
0,05	7,1%	-2,500E+15	-2,560E+14	9,765625
0,05	7,2%	-2,500E+15	-2,560E+14	9,765625
0,05	7,3%	-2,500E+15	-2,560E+14	9,765625
0,05	7,4%	-2,500E+15	-2,560E+14	9,765625
0,05	7,5%	-2,500E+15	-2,560E+14	9,765625

0,05	7,6%	-2,500E+15	-2,560E+14	9,765625
0,05	7,7%	-2,500E+15	-2,560E+14	9,765625
0,05	7,8%	-2,500E+15	-2,560E+14	9,765625
0,05	7,9%	-2,500E+15	-2,560E+14	9,765625
0,05	8,0%	-2,500E+15	-2,560E+14	9,765625
0,05	8,1%	-2,500E+15	-2,560E+14	9,765625
0,05	8,2%	-2,500E+15	-2,560E+14	9,765625
0,05	8,3%	-2,500E+15	-2,560E+14	9,765625
0,05	8,4%	-2,500E+15	-2,560E+14	9,765625
0,05	8,5%	-2,500E+15	-2,560E+14	9,765625
0,05	8,6%	-2,500E+15	-2,560E+14	9,765625
0,05	8,7%	-2,500E+15	-2,560E+14	9,765625
0,05	8,8%	-2,500E+15	-2,560E+14	9,765625
0,05	8,9%	-2,500E+15	-2,560E+14	9,765625
0,05	9,0%	-2,500E+15	-2,560E+14	9,765625

Note: This table presents expected loss values for two clients with different initial wealth levels under the same portfolio configuration. The first column shows the fixed allocation to risky assets (5%). The second column displays annual consumption as a percentage of initial wealth, ranging from 0.1% to 9% in increments of 0.1%. Columns three and four report the expected loss values for Client 2 (initial wealth = 100M) and Client 1 (initial wealth = 32M), respectively. Expected loss is computed as the average value of the quadratic loss function over 10,000 Monte Carlo simulations. The quadratic loss penalizes deviations of actual terminal wealth at time T (year 50) from a predefined bequest target, set at 50% of initial wealth. The fifth column shows the ratio of expected loss for Client 2 divided by the expected loss for Client 1, which remains constant across all simulations due to the proportional scaling of the loss function. While this table only displays results for a 5% allocation to risky assets, simulations were conducted for all allocations from 5% to 100% in 5% increments.

Appendix II

Different Beta & Gamma Values, Optimal CRRA results

Gamma	Beta	Optimal Weight to Risky Assets	Optimal Consumption (% of initial wealth)	Optimal Average CRRA Values
1.5	0.95	0.3	672,000 (2.1%)	-0.043094057
1.5	0.96	0.3	672,000 (2.1%)	-0.05145423
1.5	0.97	0.3	672,000 (2.1%)	-0.062515492
1.5	0.98	0.3	672,000 (2.1%)	-0.077376186
1.5	0.99	0.3	672,000 (2.1%)	-0.097630208
2	0.95	0.25	608,000 (1.9%)	-2.89259E-05
2	0.96	0.25	608,000 (1.9%)	-3.44812E-05
2	0.97	0.25	608,000 (1.9%)	-4.1807E-05
2	0.98	0.25	608,000 (1.9%)	-5.16134E-05
2	0.99	0.25	608,000 (1.9%)	-6.49266E-05
2.5	0.95	0.25	608,000 (1.9%)	-2.52429E-08
2.5	0.96	0.25	608,000 (1.9%)	-3.03398E-08
2.5	0.97	0.25	608,000 (1.9%)	-3.71786E-08
2.5	0.98	0.25	608,000 (1.9%)	-4.65111E-08
2.5	0.99	0.25	608,000 (1.9%)	-5.94487E-08
3	0.95	0.2	544,000 (1.7%)	-2.96318E-11
3	0.96	0.2	544,000 (1.7%)	-3.5283E-11
3	0.97	0.2	544,000 (1.7%)	-4.27171E-11
3	0.98	0.2	544,000 (1.7%)	-5.26406E-11
3	0.99	0.2	544,000 (1.7%)	-6.60712E-11
3.5	0.95	0.2	544,000 (1.7%)	-3.21404E-14
3.5	0.96	0.2	544,000 (1.7%)	-9.14154E-14
3.5	0.97	0.2	544,000 (1.7%)	-1.10676E-13
3.5	0.98	0.2	544,000 (1.7%)	-1.36386E-13
3.5	0.99	0.2	544,000 (1.7%)	-7.16662E-14
4	0.95	0.2	544,000 (1.7%)	-3.63142E-17
4	0.96	0.2	544,000 (1.7%)	-1.22934E-16
4	0.97	0.2	544,000 (1.7%)	-1.48835E-16
4	0.98	0.2	544,000 (1.7%)	-1.8341E-16
4	0.99	0.2	544,000 (1.7%)	-8.09756E-17
4.5	0.95	0.2	544,000 (1.7%)	-4.22027E-20
4.5	0.96	0.2	544,000 (1.7%)	-1.70044E-19
4.5	0.97	0.2	544,000 (1.7%)	-6.0842E-20
4.5	0.98	0.2	544,000 (1.7%)	-2.53694E-19
4.5	0.99	0.2	544,000 (1.7%)	-9.41119E-20
5	0.95	0.2	544,000 (1.7%)	-5.00689E-23

5	0.96	0.2	544,000 (1.7%)	-2.40106E-22
5	0.97	0.2	544,000 (1.7%)	-2.90694E-22
5	0.98	0.2	544,000 (1.7%)	-3.58222E-22
5	0.99	0.2	544,000 (1.7%)	-1.11666E-22
10	0.95	0.2	544,000 (1.7%)	-1.44068E-51
10	0.96	0.2	544,000 (1.7%)	-1.71543E-51
10	0.97	0.2	544,000 (1.7%)	-2.07686E-51
10	0.98	0.2	544,000 (1.7%)	-2.55931E-51
10	0.99	0.2	544,000 (1.7%)	-3.21227E-51
15	0.95	0.15	480,000 (1.5%)	-3.63476E-80
15	0.96	0.15	480,000 (1.5%)	-4.32795E-80
15	0.97	0.15	480,000 (1.5%)	-5.23981E-80
15	0.98	0.15	480,000 (1.5%)	-6.45702E-80
15	0.99	0.15	480,000 (1.5%)	-8.10439E-80
30	0.95	0.15	480,000 (1.5%)	-1.0607E-165
30	0.96	0.15	480,000 (1.5%)	-1.263E-165
30	0.97	0.15	480,000 (1.5%)	-1.5291E-165
30	0.98	0.15	480,000 (1.5%)	-1.8843E-165
30	0.99	0.15	480,000 (1.5%)	-2.365E-165

Note: The first column shows different gamma values, reflecting an investor's risk aversion. The higher the gamma, the more risk averse the investor is. The Beta column presents different beta values ranging from 0.95 to 0.99, with values closer to 1 indicating a stronger preference for future consumption. The third and fourth columns show the optimal weight of risky assets and the optimal annual consumption (as a percentage of initial wealth) for different combinations of gamma and beta values. The last column shows the optimal average CRRA values which reflects an investor's utility after choosing the optimal combination for each gamma-beta profile.