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Dynamic hedging of swaptions

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Abstract

This thesis shows that strictly following the Black model exposes the user to unexpected risk when hedging swaptions. The results emphasize that the strike offset and time to expiry have explanatory power for the hedging performance of the Black model. Furthermore, this thesis examines the impact of the volatility misestimation, arising due to the underlying assumption of constant volatility in the Black model, on delta hedging errors from a dynamic hedging strategy of swaptions. For this purpose, different selection criteria for the volatility are employed. The results from the hedging performance show that choosing a volatility that corresponds to the market implied volatility for the swaptions hedged is superior to a selection criterion that not violates the assumption of the model. Further, the thesis concludes that choosing an implied volatility that corresponds to a more liquid swaption improves the hedging performance. Also, the thesis examines whether the hedging performance can be improved by employing a model that has a more accurate volatility structure. For this purpose the constant elasticity of variance (CEV) model is employed. The results indicate that by employing a model of a different diffusion class, the hedging performance of the dynamic hedging study can be improved.

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1 Introduction

1.1 Background

We live in a risky world. Identifying and mitigating risk is a central function to increase returns. In today´s world of finance, risk has been highly underestimated during the recent boom years. With better risk measures and a more precautionary attitude to risk, the financial crisis could have been dampened. Therefore, it is highly important to have good routines for assessing and managing different aspects of risk. In the financial world, the risk arising with interest rate movements has a central function. In order to protect an investment or a loan from interest movements, one can hedge the position by using interest rate swaps, i.e. changing interest payments with a counterparty. To only protect a position from unfavourable movements, one could instead enter an option on the possibility to enter the swap in the future. Options on swaps (swaptions) are one of the most difficult derivative securities to value and hedge since the value is influenced by several properties of the underlying swap. When assessing interest rate risk, different partial derivatives are used to determine the value of the risk related to an existing situation. These derivatives have their own specific characteristics, which make them particularly challenging to understand.

This thesis will investigate the Black model (1976)[1] for hedging swaptions. This model values European interest rate options[2] and is a modification of the famous Black-Scholes model[3] for valuing equity options and it has been widely used in the financial markets for protection against unfavourable interest rate movements. The results by using the model differ from the real world, due to simplifying assumptions of the model. In practice, more advanced models for pricing and hedging options are employed, but many of the key insights provided by the model have become part of the market conventions and more complex models are built on the Black-Scholes model framework.

[1] See Black, Fischer, 1976, The pricing of commodity contracts.
[2] An option that can’t be exercised before the expiry date.
[3] See Black, Fischer and Myron Scholes, 1973, The pricing of options and corporate liabilities.

1.2 Purpose and research questions

The purpose of this thesis is to investigate the hedging performance of the Black model for swaptions based on a discrete dynamic hedging strategy. Before the hedging study is carried out, a section of the basic instruments and concepts will be provided for a deeper understanding of the underlying securities and the partial derivatives.

The major limitations of the Black model concern the assumptions of a log-normal distribution[4] of the forward swap rate and a constant volatility across different strikes and time to expiry of the underlying security. In a typical market, the volatility structure is not constant.

This thesis examines the impact of the volatility misestimation on delta hedging errors and examines scenarios where the swaptions are hedged at different volatilities. Within the Black model framework, three different selection criteria for the volatility are tested for. Furthermore, this thesis examines whether employing a model of an alternative diffusion class could mitigate the volatility misestimation. More specifically, a model of constant elasticity of variance (CEV) model[5] is employed.

In the light of the previous discussion, the thesis aims to focus on answering the following two research questions:

1. Can the hedging performance be improved by changing the selection criterion for the volatility in the Black model?
2. Can the hedging performance be improved by employing a model that has a more accurate volatility structure?

1.3 Disposition

The thesis is structured as following. In part 2, the basic securities that are traded in the financial markets and terminology are introduced. It should be of great importance to get a thorough understanding of these securities before employing any model to hedge derivatives. Part 3 provides the theoretical background of swaption pricing. In part 4, partial derivatives are introduced and the concept of hedging as well as a general hedging strategy for swaptions is presented. Part 5 describes the step-by-

[4] A variable is said to follow a log-normal distribution if the natural logarithm of that variable follows a normal distribution.

[5] See Cox, John C. and Stephen A. Ross, 1976, The Valuation of Options for Alternative Stochastic Processes, pp. 145-166.

step methodology used in the hedging study to examine the performance of the models. In part 6, a description of the data is provided. In part 7, the results obtained from the hedging study are presented along with a critical and in-depth analysis. In order to provide a more thorough understanding, this part will also provide a section dealing with the hedging characteristics. In part 8, the reached conclusions on the research questions and all major findings are summarized. Finally, part 9 provides suggestions for further research.

2 Basic instruments and terminology

This part provides an introduction to the basic instruments and terminology that will lay the basis for the progress of this thesis.

2.1 LIBOR

The London Interbank Offer Rate (*LIBOR*) is widely used as the underlying in interest rate forwards, swaps and swaptions and is compiled by the British Banker's Association in association with Reuters. The rate is based on the 1-month, 3-month, 6-month and 12-month interest rates at which banks borrow unsecured funds from each other in the London interbank market. The importance of the *LIBOR* is due to several reasons. First, it is a truly international reference rate and it has been established in the market for a long time. The banks, represented when computing it, have the highest credit ratings and are also the most active in the cash market. The London base is dominant as more than 20% of all international lending and 30% of all foreign exchange transactions occurs in the London market.[6] In this thesis, the *LIBOR* will be denoted L.

2.2 Accrual factor

The amount of interest that will be received on a deposit is calculated by multiplying the *LIBOR* by an accrual factor, defined as the amount of time that the deposit has been in place. The accrual factor is computed by dividing the number of days in the deposit period by the number of days in a year. Different market conventions are used in different markets, e.g. actual/360 and actual/365. The conventions are computed by taking the actual number of days in the period divided by a base. In this thesis, the accrual factor will be denoted α. For ease of the notation, α is assumed to be the same for all

[6] British Banker's Association.

periods. A deposit of a notional amount N at *LIBOR* will therefore, according to the terminology, yield a payout at expiry of $N\alpha L$. This is illustrated in figure 1.

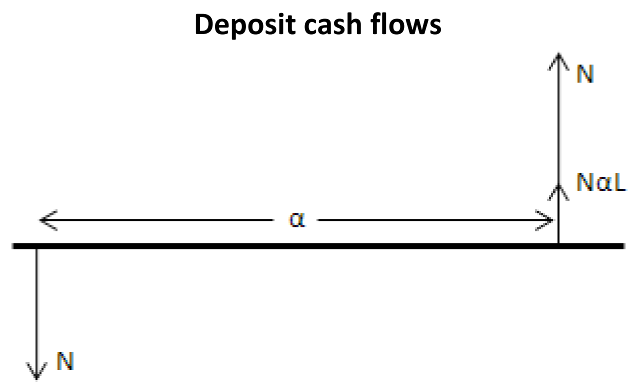

Figure 1: The horizontal axis represents the time of cash flows. The received cash flows are the ones above the axis and those paid are below.

2.3 Zero coupon bond

A *zero coupon bond* (ZCB) is bought at a price lower than its notional and the notional amount is repaid at time of expiry. This kind of bond does not pay any coupons, hence the term *zero coupon bond*. They can be long term investments, typically with ten years expiry, and can be held until expiry or sold on secondary bond markets. In the hedging procedure that will be introduced later, $ZCBs$ will be an important factor of the constructed portfolio. The value of a one year ZCB is the notional multiplied by the one year spot rate discounted until today. This implies that $ZCBs$ could be employed as discount factors. For the models introduced later, the following discrete date structure is introduced.

$$T_0 < T_1 < T_2 \dots T_{n-1} < T_n$$

A ZCB bought at T_0 with expiry T_1will be denoted $D_{T_0 T_1}$ and is to be used interchangeable with the discount factor from T_0 to T_1.

2.4 FRA

For the development of this section, a point in time t is introduced. This t is located at an unspecified date prior to T_0 on the date structure introduced in section 2.3, i.e. $t < T_0$. A forward rate agreement, *FRA*, is an over-the-counter (OTC) derivative contract between two counterparties to exchange cash payments in the future. The *FRA* contract specifies the interest rate to be exchanged, the reset date T_0, the payment date T_1 and the notional amount. In general, one party of a *FRA* contract is paying the counterparty an amount $N\alpha K$, where K is the fixed rate. In return, an amount of $N\alpha L_{T_0}[T_0, T_1]$ will be received from the counterparty, where $L_{T_0}[T_0, T_1]$ is the floating rate (mostly *LIBOR*) for the period $[T_0, T_1]$ that sets on date T_0. This is illustrated in figure 2.

Cash flows for the holder of an *FRA*

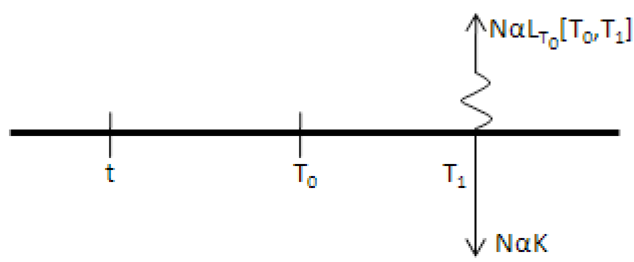

Figure 2: The horizontal axis represents the time of cash flows. The wavy cash flow is the cash flow from the floating rate, which is not known at t. The cash flow below the axis represents the cash flow from the fixed rate, which is known at t.

The party that has agreed to pay a fixed interest rate, wishing to protect itself against a potential future rise in the reference rate is known as the buyer of the *FRA*, while the party agreed to receive fixed interest rate, wishing to protect itself against a future drop in the reference rate is known as the seller of the *FRA*. Within a *FRA* contract, payments are calculated on the notional over the period and only the net difference is paid out. The exposure to both parties is therefore only the difference between the rate agreed on and the actual settlement rate (*LIBOR*) at T_0. *FRA* contracts provide a possibility for investors to lock in a prevailing interest rate for a specific time interval in the future, without facing the risk of unfavourable market movements. A purpose of the *FRA* contract is to fix today the amount of interest an investor will receive on a deposit he intends to make at some future date. Suppose the investor wishes to invest a unit amount from time T_0 until T_1. Instead of waiting until T_0, when market conditions may have changed unfavourably, the investor can fix today the amount of interest received. To illustrate this, a trade is set up and the resulting cash flows to the counterparty are studied. To fix the amount of interest received at time t, the investor needs to sell a *FRA* contract at $t < T_0$, at the prevailing forward *LIBOR* $L_t[T_0, T_1]$. At time T_0, the unit capital is invested at the then spot *LIBOR* $L_{T_0}[T_0, T_1]$ and at time T_1 the capital is received with interest $\alpha L_{T_0}[T_0, T_1]$. Under the *FRA*, the investor will also receive $\alpha(L_t[T_0, T_1] - L_{T_0}[T_0, T_1])$. The net income at time T_1 will be the notional plus exactly

$\alpha L_t[T_0, T_1]$, which is the amount locked in when the *FRA* was entered. If the investor instead wishes to fix today the interest that will be paid in the future, a *FRA* contract is bought and the net cash flow from the *FRA* is $\alpha(L_{T_0}[T_0, T_1] - L_t[T_0, T_1])$.

2.5 Forward LIBOR

When two counterparties enter into a *FRA*, typically both parties do so at zero cost at time t. The forward *LIBOR*, $L_t[T_0, T_1]$, is defined as the rate that makes the value of the *FRA* equal to zero for the period $[T_0, T_1]$. It is synonymous to the fixed interest rate K that gives both legs of the agreement a breakeven value. However, it should be noted that the value of the *FRA* changes, although zero when entered, as the rate fluctuates over time. Before going any further, recall the discount factor introduced in section 2.3. For any $T_i \geq t$, a discount factor D_{tT_i} is defined as the value at t of a ZCB that yields one unit of cash at time T_i. The *LIBOR* rate at $T_0$for the period $T_0$to T_1can be rewritten as a fraction of the discount factors.

$$L_{T_0}[T_0, T_1] = (D_{T_0T_0} - D_{T_0T_1})/\alpha D_{T_0T_1} \quad (1)$$

Equation (1) states that the final payment of a *FRA* depends on two zero coupon bonds with expiry at T_0 and T_1respectively. Thus, the *FRA* is a derivative of these bonds. To value the *FRA* and in order to determine the forward *LIBOR* when the contract is originated, a replicating portfolio of two zero coupon bonds can be set up. For this purpose, recall that the net payment when buying a *FR*A is equal to $\alpha(L_{T_0}[T_0, T_1] - K)$. First, assume that one has an amount of cash at t equal to

$$V_t = D_{tT_0} - (1 + \alpha K)D_{tT_1} \quad (2)$$

In order to replicate the cash flows of the *FRA*, one unit ZCB with expiry at T_0 is bought and $(1 + \alpha K)$ units ZCB with expiry at $T_1$are sold at t. At T_0, the payment from the ZCB that expires is received and this amount is deposited until T_1. At T_1, the deposit and the outstanding ZCB expire, which gives a net payment of

$$\left(1 + \alpha L_{T_0}[T_0, T_1]\right) - (1 + \alpha K) = \alpha\left(L_{T_0}[T_0, T_1] - K\right)$$

which is exactly the net payment received under the *FRA*. Hence, the trade in the two $ZCBs$ replicates the *FRA*. The value of the *FRA* is therefore equal to V_t, which in general is not zero. As the fixed rate *K* is equal to $L_t[T_0, T_1]$, the value of the fixed rate that gives the *FRA* a breakeven value at t is obtained by setting equation (2) to zero and solving for the fixed rate. The fixed rate is therefore given as

$$L_t[T_0, T_1] = (D_{tT_0} - D_{tT_1})/\alpha D_{tT_1}$$

By using this expression, the valuation formula can be rewritten as

$$V_t = \alpha D_{tT_1}(L_t[T_0, T_1] - K)$$

2.6 Interest rate swaps

An interest rate swap (abbreviated swap) is an agreement between two parties under which they agree to exchange a predetermined fixed interest rate for a floating interest rate (mostly *LIBOR*) periodically at specific time intervals, based on a notional amount and for an agreed period of time. The main swap product in the market is the plain vanilla swap, also known as a generic swap, typically constructed as an exchange of fixed interest rate obligations for floating rate obligations. This type of interest rate swap is equivalent to a portfolio of forward rate agreements between the two parties. Whereas a *FRA* only involve one exchange of future cash flows, the swap generally involve several future exchanges. The party obligated to pay fixed and receive floating interest payments is the payer, while the part that pays floating and receives fixed payments is the receiver. One could think of a swap as a trade where the payer is short in a fixed rate bond and long in a floating rate bond, while the receiver is long in a fixed rate bond and short in a floating rate bond. Subsequently, the value of the swap can be written as $V_{P.Swap} = V_{floating\ b.} - V_{fixed\ b.}$ from the position of the payer or $V_{R.swap} = -V_{P.Swap}$ from the position of the receiver.

In the fixed leg of the swap, all cash flows are known when the swap is originated. The cash flow at T_i is given by αK, where α is the accrual factor for the period $[T_{i-1}, T_i]$. In the floating leg of the swap, the cash flows will be determined during the life of the swap. On the payment dates, starting at T_1, the cash flow at T_i is determined at T_{i-1} and the accrual factor α is multiplied by $L_{T_{i-1}}[T_{i-1}, T_i]$ which is the *LIBOR* for the period $[T_{i-1}, T_i]$. For each cash flow exchange, the floating rate is determined at the prevailing reference rate level in the beginning of each time interval, but exchanged at the end of each time interval.

Swaps are a useful tool when managing the risk of an investment. They are important in financial management because they make it possible to hedge the interest rate risk. In figure 3, data of the global OTC derivatives market is presented. The market is dominated by interest rate contracts, which has increased a lot the recent years. The interest rate contracts are divided into *FRAs*, swaps and options (including swaptions). Swaps contribute by about 90 % of the total market value for these types of contracts.[7]

The global OTC derivatives market

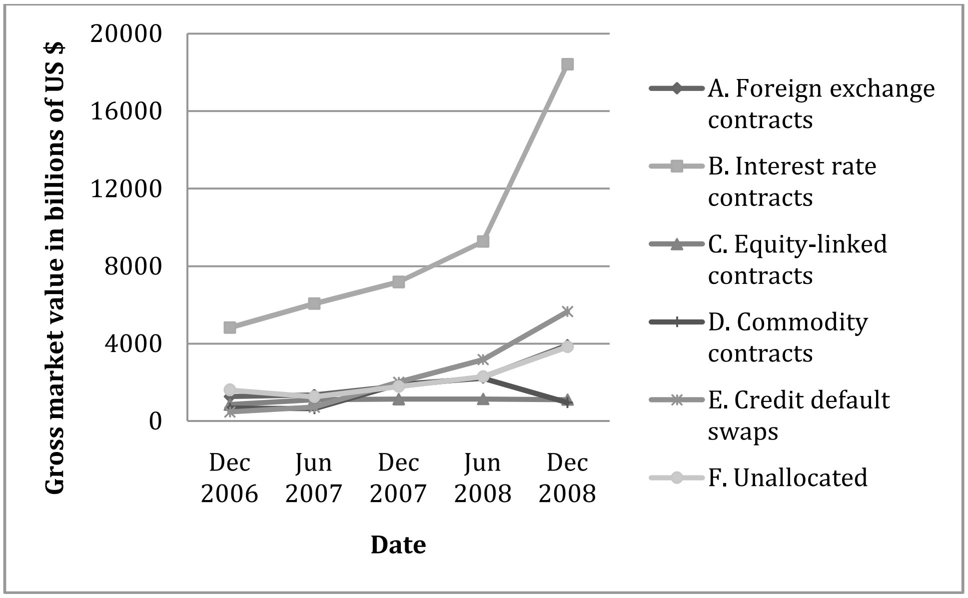

Figure 3: The gross market value is plotted for different contracts during the period December 2006 to December 2008.
Source: Bank for International Settlements.

2.7 Forward swaps

A forward swap is an agreement at time t to enter a swap at T_0 in the future at a predetermined price. If future income or payment streams are known, a forward swap could be an effective hedge instrument and it could offer a higher degree of flexibility in liability management. It could be lucrative for the investor or the company not having to enter the capital market when not favourable. Generally, it is priced through a combination of two swaps of different durations in order to meet the certain time frame that an investor needs. The basic rationale is that a one-year and a three-year swap could be entered into, in order to create a two-year swap starting in one year. The main reason why investors use forward swaps is that it becomes possible to expand the maturity of existing fixed rate debt. One can also lock in a financing rate on an anticipated debt. Forward swaps make it also possible to shorter the maturity structure of long term fixed rate debt and reduce sensitivities in a portfolio of debt.

[7] Bank for International Settlements.

2.8 Forward swap rate

The forward swap rate is the fixed swap rate K, agreed on between the counterparties when entering a forward swap, which equal the present value of the fixed and the floating leg of the forward swap. In other words, the fixed swap rate that makes the present swap value equal to zero. It is the rate agreed on at present, t, for a swap that starts at a predetermined date in the future, T_0, and that makes fixed payments on dates $T_1, \dots, T_n$. In order to value a forward swap and resolve for the forward swap rate that gives the fix and the floating leg a breakeven value when originated, the value of the fixed rate leg and the floating rate leg are studied separately. The fixed rate leg consists of a series of payments on predetermined dates $T_1, T_2 \dots, T_n$. The payment at time T_i is αK and the total value is therefore given by

$$V_t^{FXD} = K\alpha \sum_{i=1}^{n} D_{tT_i} = K \cdot PVBP_t$$

$\alpha \sum_{i=1}^{n} D_{tT_i}$ is the present value of a basis point, also denoted $PVBP_t$, and represents the value of the fixed leg if the fixed rate were unity. For the progress of this thesis, the two expressions for the present value of a basis point will be used interchangeably. The value of the floating leg can be determined by setting up a trading strategy that replicates the floating payments. At t, suppose one has an amount of cash

$$V_t^{FLT} = D_{tT_0} - D_{tT_n}$$

From this cash, one ZCB with expiry at $T_0$is bought and one ZCB with expiry at T_n is sold. At T_0, the unit paid by the ZCB is deposited at *LIBOR* until T_1. At T_1, an amount of $1 + \alpha L_{T_0}[T_0, T_1]$ is received. The term $\alpha L_{T_0}[T_0, T_1]$ is the floating payment one needs to replicate the swap and the extra unit amount of notional is deposited at *LIBOR* until T_2. This is continued until $T_n$is reached, where an amount of $1 + \alpha L_{T_n}[T_{n-1}, T_n]$ is received. The *LIBOR* part of this payment is the amount needed for the last floating payment of the swap and the notional amount pays the amount owed on the ZCB sold at t. The net value of a payers swap at t is

$$V_{P.Swap;\,t} = V_t^{FLT} - V_t^{FXD} = D_{tT_0} - D_{tT_n} - K \cdot PVBP_t \tag{3}$$

The value of K which sets $V_{P.Swap;\,t}$ to zero is the forward swap rate FS_t. Setting $V_{P.Swap;\,t} = 0$ in equation (3) yields

$$FS_t = \frac{D_{tT_0} - D_{tT_n}}{PVBP_t} \tag{4}$$

If this is substituted back into equation (3), one gets the following expression for the value of a payer forward swap

$$V_{P.Swap;\,t} = PVBP_t(FS_t - K) = \alpha \sum_{i=1}^{n} D_{tT_i}\,(FS_t - K) \tag{5}$$

Analogously, a receiver forward swap is worth $-V_{P.Swap;\,t}$.

2.9 Swaptions

A swaption or a swap option is an option, where a forward swap is the underlying security. A swaption entitles the holder of the security to enter a forward swap at a predetermined time and interest rate in the future. In a swaption contract, the holder has an option, but not the obligation, to exchange future cash flows with the counterparty of the trade. Like traditional swaps, swaptions are commonly used hedging instruments against interest rate fluctuations. In comparison to the swap, the holder of a swaption contract is not enforced to enter the underlying forward swap contract. The swaption provides the holder protection from unfavourable interest rate movements, leaving the investor with the possibility to profit from favourable movements. When two parties enter into a swaption contract they must agree on the starting date, the expiry of the swaption and all the specific details of the underlying forward swap. The strike of the swaption is the fixed interest rate agreed on between the counterparties. Swaption contracts exist as both calls and puts. A call swaption gives the holder the right, but not the obligation, to pay a fixed rate equal to the strike of the forward swap and receive the floating interest rate agreed on (mostly *LIBOR*), whilst the holder of a put swaption has the right, but not the obligation, to receive fixed rate interest at the strike rate and pay floating rate interest. In line with the terminology introduced on swaps, a call swaption is also denoted a payer swaption, $V_{P.Swaption}$, whilst a put swaption is denoted a receiver swaption, $V_{R.Swaption}$.

In this thesis, empirical data from the swaption market is investigated and different hedging strategies will be compared. The hedging study will use swaptions with expiry at T_0, on a forward swap contract in the period T_0 to T_n. Consider the case where a firm has known future yearly floating interest rate payments in the period T_0 to T_n. To protect itself from rising interest rates the firm could purchase a payer swaption. By paying a premium, the firm is able to receive floating interest payments and pay fixed interest rate during the forward swap period. At the expiry date of the swaption, i.e. $T_0 - t$ years from now, there are two possible outcomes. If the market swap rate is higher than the predetermined fixed rate, the swaption is exercised and the firm can make their committed floating interest rate payments at a lower rate than the market interest rate. This is profitable for the firm. If the market swap rate is lower than the strike rate, the swaption is not exercised and the firm will use the lower interest rates in the market. The payoff of a payer swaption is illustrated in figure 4.

Payer swaption payoff at expiry

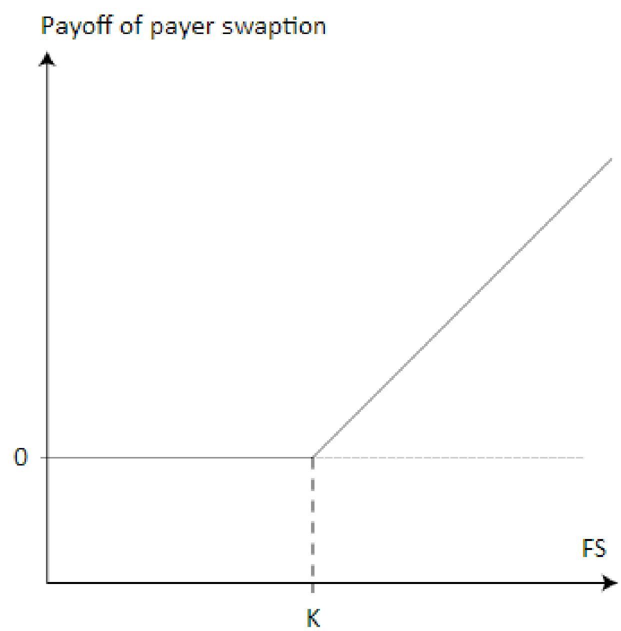

Figure 4: The payoff from a payer swaption at expiry is increasingly positive with the forward swap rate, above the predetermined strike rate.

2.10 The swaption market

The participants in the swaption market are predominantly large corporations, banks, financial institutions and hedge funds. Major investment and commercial banks make markets for swaptions and also trade in the swaption interbank market. Typically, the market makers manage large portfolios of swaptions, written with various counterparties. Swaption markets exist in most of the major currencies in the world, where the largest are in US Dollars, Euro, Sterling and Japanese Yen. The swaption market is an OTC market, i.e. not traded on any exchange. Legally, a swaption is an agreement between two counterparties to exchange the required payments. The counterparties are exposed to each others' failure to make scheduled payments on the underlying forward swap, although this exposure is typically mitigated through the use of collateral agreements, whereby a margin is posted to cover the anticipated future exposure.

3. Theoretical background of swaption pricing

This part introduces the valuation models employed in the hedging study that will be carried out.

3.1 Swaption pricing

The aim in this section is to present the fundamental ideas behind pricing and hedging swaptions without digging to deep into the technicalities.[8] First, the concept of numeraires is introduced. Numeraires plays an important role in the theory of option pricing and the term represents a unit of account. Generally, the numeraire is applied to a single good, which becomes the base good for the valuation of other goods. In the financial market, one can change the numeraire when pricing assets. This procedure will be explained and it closely follows the general framework for the change of numeraire technique introduced by Geman et al.[9] If the formula

$$M_t = exp\left(\int_0^t r_s\, ds\right)$$

is the price at t of \$1, invested in the money market at time 0, the well-known Black-Scholes model presented in the next section states that all assets, priced in terms of the money market, are martingales w.r.t. the risk neutral probability measure, denoted Q.[10] That is

$$\frac{S_t}{M_t} = E_Q\left[\frac{S_T}{M_T}\middle|\mathcal{F}_t\right] \qquad \forall t \leq T$$

Assume that $N_t > 0$ is another strictly positive asset and hence a martingale when priced in terms of the money market. A new probability measure, Q^N, could then defined by the Radon-Nikodym derivative

[8] For a deeper understanding of the technicalities see Hunt, P.J. and J.E. Kennedy, 2004, Financial derivatives in theory and practice, p. 149 ff.

[9] See Geman, Helyette, Nicole El Karoui and Jean-Charles Rochet, 1995, Changes of Numeraire, Changes of Probability Measure and Option Pricing, pp. 443-458.

[10] For those unfamiliar with the risk-neutral measure, this is the resulting probability measure when one assumes that the present value of financial assets is equal to the expected value of the future payoff of the asset discounted at the risk-free rate. For further details see Geman, Helyette, Nicole El Karoui and Jean-Charles Rochet, 1995, pp. 443-458.

$$\frac{\partial Q^N}{\partial Q} = \frac{M_0}{M_T}\frac{N_T}{N_0}$$

With Bayes' Theorem, one can then show that S_t is a martingale when priced in terms of the new numeraire, N_t

$$E_{Q^N}\left[\frac{S_T}{N_T}\middle|\mathcal{F}_t\right] = E_Q\left[\frac{M_0 N_T S_T}{M_T N_0 N_T}\middle|\mathcal{F}_t\right]\Big/E_Q\left[\frac{M_0 N_T}{M_T N_0}\middle|\mathcal{F}_t\right]$$

$$= \frac{M_t}{N_t}E_Q\left[\frac{S_T}{M_T}\middle|\mathcal{F}_t\right] = \frac{M_t S_t}{N_t M_t} = \frac{S_t}{N_t}$$

This framework will prove to be very helpful in order to hedge European swaptions. For the progress of this section, the valuation of the *FRA* contract is recalled. From section 2.5, the value of a *FRA* at t was given by

$$V_t = D_{tT_0} - (1+\alpha K)D_{tT_i}$$

It should be noted that a model was not specified for the evolution of asset prices in order to price the derivative. This rests on the case that it was a static replicating portfolio. For more complex derivative securities, like the swaption contract, it is however instructive to apply the concepts of numeraires and risk measures, when valuing a *FRA*. Suppose in this scenario that $\mathbb{N}$ is the general measure, N_t is the numeraire process and that $\{\mathcal{F}_t\}$ is the filtration generated by the assets in the market, the price of a *FRA* is given by[11]

$$\begin{aligned} V_t &= N_t \mathbb{E}_{\mathbb{N}}\left[\alpha\left(L_{T_0} - K\right)N_{T_1}^{-1} \,\middle|\, \mathcal{F}_t\right] \\ &= N_t \mathbb{E}_{\mathbb{N}}\left[\alpha\left(L_{T_0} - K\right)\mathbb{E}_{\mathbb{N}}\left[N_{T_1}^{-1} \,\middle|\, \mathcal{F}_{T_0}\right] \middle|\mathcal{F}_t\right] \\ &= N_t \mathbb{E}_{\mathbb{N}}\left[\alpha\left(L_{T_0} - K\right)D_{T_0 T_1} N_{T_0}^{-1} \,\middle|\, \mathcal{F}_t\right] \\ &= D_{tT_0} - (1+\alpha K)\, D_{tT_1} \end{aligned}$$

where the last step follows from substituting for L_{T_0}, using equation (1). The swap contract can be valued identically since it is only a linear combination of *FRA* contracts and $ZCBs$, as developed in section 2.8. To price the swaption contract, recall that $V_{P.Swap;\,t}$ is the value at t of a payer swap, starting at T_0 and making fixed payments at $T_1, \dots, T_n$. Since a swaption contract gives the holder an

[11] See Hunt, P.J. and J.E. Kennedy, 2004, p. 238.

option, but not the obligation, to exchange future cash flows with the counterparty of the trade, the value of payer swaption at expiry T_0 is given by

$$V_{P.Swaption;\,T_0} = {V_{P.Swap;\,T_0}}^{+}$$

The swaption value at t is then, by the valuation formula introduced when valuing the *FRA*, given by

$$V_{P.Swaption;\,t} = N_t \mathbb{E}_{\mathbb{N}} \left[{V_{P.Swap;\,T_0}}^{+} N_{T_0}^{-1} \middle| \mathcal{F}_t \right]$$

$$V_{P.Swaption;\,t} = N_t \mathbb{E}_{\mathbb{N}} \left[\alpha \sum_{i=1}^{n} D_{T_0 T_i} \left(FS_{T_0} - K \right)^{+} N_{T_0}^{-1} \middle| \mathcal{F}_t \right] \quad (6)$$

3.2 The Black-Scholes model and the Black model

This section provides a short introduction to the Black-Scholes model. Furthermore, the Black model is introduced, which is a modification of the traditional Black-Scholes model for the valuation of equity options with futures contracts as the underlying security. The traditional Black-Scholes model is a model for obtaining the price of European option contracts. The general assumptions of the Black-Scholes model are that the financial market is frictionless and that borrowing and lending of cash at a known risk-free interest rate in the money market is possible continuously. Furthermore, it assumes that there are no restrictions on short selling and that security prices follow a geometric Brownian motion with constant drift and volatility. Due to these simplifying underlying assumptions, the Black-Scholes formula is only an approximation, but is still used in the financial market.[12] The reasons are that it is easy to apply and that it explicitly models the relationship of all variables. Furthermore, it provides a basis for more refined models. As per the model definitions, one assumes that the underlying follows a geometric Brownian motion. If the underlying is a stock, denoted S, this is written as

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

under the real-world statistical measure, where μ is the drift, σ the volatility, W_t is Brownian and dW stands for the uncertainty in the price history of the underlying. With the notations introduced, the traditional Black-Scholes pricing formula for stock options takes the following form

$$V_{Call\ option} = S_t \Phi(d_1) - K \cdot D_{tT_0} \Phi(d_2)$$

[12] See Hull, John C., 2006, Options, Futures and Other Derivatives, p. 281.

$$V_{Put\ option} = K \cdot D_{tT_0}\Phi(-d_2) - S_t\Phi(-d_1)$$

$$D_{tT_0} = e^{-\mathrm{i}(T_0-t)}$$

$$d_1 = \frac{1}{\sigma\sqrt{(T_0-t)}}\left[ln\frac{S_t}{K} + \left(\frac{1}{2}\sigma^2 + i\right)(T_0 - t)\right], \qquad d_2 = d_1 - \sigma\sqrt{(T_0-t)}$$

where

$K =$ strike price of the option
$i =$ risk-free interest rate, continuously compounded
$S_t =$ spot price of the stock at time t
$\sigma =$ volatility
$T_0 - \mathrm{t} =$ time to expiry
$\Phi =$ cumulative Gaussian distribution function

$\Phi(d_1)$ and $\Phi(d_2)$ in the pricing formula are the probabilities that the option expires in-the-money. The factor $\Phi(d_2)$ is the equivalent martingale probability measure, where the corresponding numeraire is the risk-free asset. The equivalent martingale measure is the one referred to in section 3.1 as the risk neutral probability measure.

The main characteristic in the Black model compared to the traditional Black-Scholes model is that it prices European options as if the value at T_0 depends on the future price rather than on the spot price, i.e. a futures contract is the underlying security. In addition to the assumptions stated for the traditional Black-Scholes model, the assumption of a log-normal distribution of the future price of the underlying security is central for the Black modification of the formula. With this assumption stated, further assumptions of a geometric Brownian motion for the evolution of the underlying security and its forward price does not have to be stated.[13] The elegance of the Black model has its own limitations however. The assumption of constant volatility and risk-free interest rates limits its applicability and the model is only valid for valuation of European options.[14] The Black model extension from the traditional Black-Scholes model is nevertheless widely accepted in the financial markets for valuation of European

[13] See Hull, John C., 2006, p. 613.
[14] See Akume, Daniel, Bernd Luderer and Gerhard-Wilhelm Weber, 2003, Pricing and hedging of swaptions, p. 6.

interest rate options.[15] One reason is that the volatility of the underlying security, which usually is computed from market data, is the only value that has to be computed for input in the model.

3.3 Swaption pricing with the Black model

This section provides an introduction of the forward swap measure and how to apply the Black framework in the valuation of swaptions. If the forward swap rate is assumed log-normal under the forward swap measure[16], swaptions can be priced by the Black model. The forward swap measure, $\mathbb{S}$, used in the Black model for pricing and hedging swaptions is the one induced by selecting the $PVBP$ as the numeraire asset. Recall the discussion of numeraires developed under section 3.1. Under this probability measure, all assets discounted by the $PVBP$ will be martingales.[17] The notation $\mathbb{E}_{\mathbb{S}}$ denotes the expectation in a world that is forward risk neutral w.r.t. the $PVBP$. Under this condition, equation (6) takes the following form

$$V_{P.Swaption;\,t} = \alpha \sum_{i=1}^{n} D_{tT_i}\, \mathbb{E}_{\mathbb{S}} \left[\left(FS_{T_0} - K \right)^{+} \middle| \, \mathcal{F}_t \right] \tag{7}$$

Note the difference between equations (6) and (7). In the former, the discounting is inside the expectation operator, while in the latter outside. By using a world that is forward risk neutral w.r.t. the $PVBP$, the valuation of the swaption is simplified since this implies that the forward swap rate is its expected future spot swap rate.[18] This gives that a swaption can be valued by calculating its expected payoff in a world that is forward risk neutral w.r.t. the $PVBP$. Recall from equation (4) that the forward swap rate, FS_t, that gives the forward swap a breakeven value at time T_0 is of the form

$$FS_t = \frac{D_{tT_0} - D_{tT_n}}{PVBP_t}$$

which is the ratio of the asset prices over the numeraire, and hence must be a martingale under the forward swap measure $\mathbb{S}$. By introducing σ_t as the forward swap rate volatility and W_t as a Brownian motion under $\mathbb{S}$, the forward swap rate is modeled by

$$dFS_t = \sigma_t\, FS_t dW_t$$

[15] See Akume, Daniel, Bernd Luderer and Gerhard-Wilhelm Weber, 2003, p. 5.
[16] For further details see Jamshidian, Farshid, 1997, Libor and swap market models and measures, pp. 293-330.
[17] See Barton, Geoff, Tim Dun and Erik Schlögl, 2001, Simulated swaption delta-hedging in the lognormal forward Libor model, p. 4.
[18] See Hull, John C., 2006, p. 597.

A European payer swaption can then be valued with the Black formula

$$V_{P.Swaption;\,t} = \alpha \sum_{i=1}^{n} D_{tT_i} \left[FS_t \Phi(d_1) - K\Phi(d_2) \right] \tag{8}$$

Analogously, a European receiver swaption is valued as

$$V_{R.Swaption;\,t} = \alpha \sum_{i=1}^{n} D_{tT_i} \left[K\Phi(-d_2) - FS_t \Phi(-d_1) \right]$$

where

$$d_1 = \frac{\ln(FS_t/K)}{\sigma_t \sqrt{T_0 - t}} + \frac{1}{2}\sigma_t \sqrt{T_0 - t},$$

$$d_2 = \frac{\ln(FS_t/K)}{\sigma_t \sqrt{T_0 - t}} - \frac{1}{2}\sigma_t \sqrt{T_0 - t} = d_1 - \frac{1}{2}\sigma_t \sqrt{T_0 - t}$$

$$\sigma_t^{\,2} = \frac{1}{T_0 - t} \int_t^{T_0} \sigma_u^{\,2} du$$

One limitation of the Black model is the assumption of log-normal distribution of the forward swap rate under the forward swap measure. Usually, the forward swap rate seldom follows a log-normal distribution.[19] The market volatility normally fluctuates in two dimensions, the strike rate and time to expiry. Basically this means that when changing σ_t with the strike and time to expiry, a different model is employed for each strike and expiry date. This implies difficulties when managing large books of swaptions. The delta and vega risks, as developed for later, calculated at a given strike may not be consistent with the same risks calculated at other strikes, which put in uncertainty into the hedging of risks across strikes. Furthermore, if σ_t varies with the strike, one can expect that σ_t also varies systematically as the forward rate changes.[20] Any vega risk from this change could be hedged more appropriately as delta risk. Another approach is to apply an alternative diffusion model that can price swaptions with different strikes and expiry dates more properly.

[19] See Hagan, Patrick S. and Diana E. Woodward, 1998, Equivalent Black Volatilities, p. 1.
[20] See Dupire, Bruno, 1993, Pricing and Hedging with Smiles, and Dupire, Bruno, 1994, Pricing with a smile, pp. 18-20.

3.4 Beyond the Black model

In this section, a model with a different diffusion class is introduced and an approximation for the computation of the implied volatility in this model will be derived. In the Black formula, one presumes that the forward rate FS_t is log-normally distributed under the forward swap measure $\mathbb{S}$ and modeled as a log-normal martingale by

$$dFS_t = \sigma_t \, FS_t dW_t$$

For the development of more accurate models, recall the fundamental ideas behind pricing and hedging swaptions, developed in section 3.1. Generally these models are of the form

$$dFS_t = \alpha_t A(FS_t) dW_t$$

under the forward swap measure induced by choosing the $PVBP$ as the numeraire. The value of a payer swaption under this measure is, as presented in section 3.3, given by the expected value of the swaption

$$V_{P.Swaption;\,t} = \alpha \sum_{i=1}^{n} D_{tT_i} \, \mathbb{E}_{\mathbb{S}} \left[\left(FS_{T_0} - K \right)^+ \middle| \, \mathcal{F}_t \right]$$

With singular perturbation techniques[21], these models can be analyzed and explicit algebraic formulas for the value of European swaptions can be found.[22] From these expressions, the implied volatilities of the swaptions can be obtained by

$$\sigma_t = \frac{a|A_{f_{av}}|}{f_{av}} \left\{ \begin{matrix} 1 + \frac{1}{24}\left[\frac{A''}{A} - 2\left(\frac{A'}{A}\right)^2 + \frac{2}{f_{av}^2}\right](FS_t - K)^2 + \\ \frac{1}{24}\left[2\frac{A''}{A} - \left(\frac{A'}{A}\right)^2 + \frac{1}{f_{av}^2}\right] a^2 {A^2}_{f_{av}} (T_0 - t) + \ldots \end{matrix} \right\} \quad (9\text{ a})$$

where

$$f_{av} = \frac{1}{2}(FS_t + K) \quad (9\text{ b})$$

This denotes that A and its derivatives are estimated halfway between today's forward swap rate and the strike rate. Furthermore, a is the "sum-of-squares average" of α_t.

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[21] For further details see Cole, J. D. and J. Kevorkian, 1981, Perturbation Methods in Applied Mathematics.

[22] See Hagan, Patrick S. and Diana E. Woodward, 1998, p. 1.

$$a = \left(\frac{1}{(T_0 - t)}\int_t^{T_0} \alpha^2{}_u du\right)^{1/2} \quad (9\text{ c})$$

This framework will prove to be very useful for the progress of this thesis since it provides an easy and fast way of applying a more complex model.

3.4.1 The CEV model

In this section, an extension of the Black model is introduced in order to investigate whether any potential improvements in terms of hedging swaptions can be achieved by employing a model that has a more accurate volatility structure. More specifically, a constant elasticity of variance (CEV) model is introduced. The general CEV model is a generalization of the traditional Black-Scholes model and is of the following diffusion class

$$dS_t = \mu S_t dt + \sigma {S_t}^{\psi/2} dW_t$$

where the constant parameter ψ represents the elasticity of the instantaneous variance. The instantaneous variance of the percentage price change is equal to $\sigma^2 / S^{2-\psi}$, i.e. a direct inverse function of the stock price. However, in the Black-Scholes model that corresponds to the restraining special case of $\psi = 2$, the variance rate is not the function of the stock price. Both economic rationale and empirical studies suggests that an option pricing formula based on constant elasticity of variance diffusion can fit market prices more properly than one based on the Black-Scholes model.[23] The progress of this thesis aims at evaluating two cases of the constant elasticity of variance class model, $\psi = 1$ and $\psi = 0$, in order to make a comparison with the Black model under the special case. Onwards, the elasticity case chosen will be denoted by the parameter β, where $\beta = \psi/2$.

By employing a CEV diffusion class model, the flat volatility structure suggested by the Black model can be more closely fitted to the market volatility. The impact that the parameter β in the CEV model has on the volatility structure is given by figure 5. From this figure, it follows that a lower β gives more slope to the volatility structure.

[23] See Beckers, Stan, 1980, The Constant Elasticity of Variance Model and Its Implications for Option Pricing, p. 661.

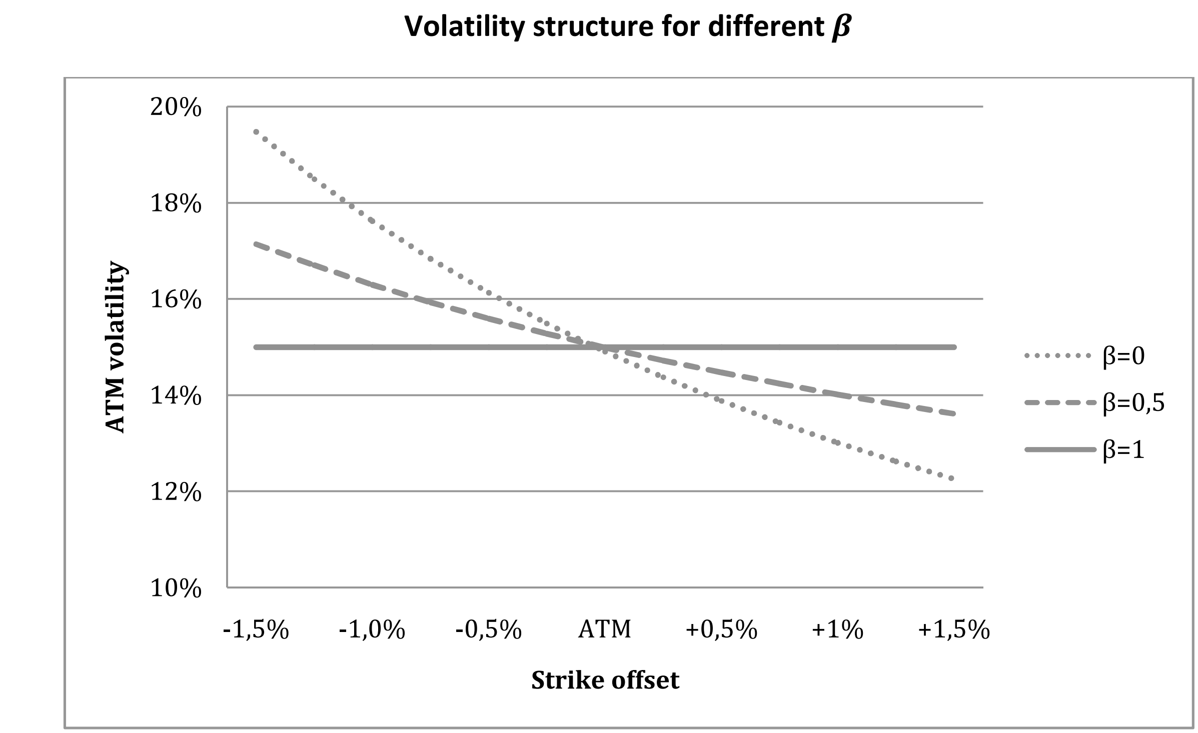

Figure 5: The *ATM* volatility for three different β is plotted w.r.t. the strike offset.

3.4.2 Swaption pricing with the CEV model

In this section, the procedure of pricing swaptions with the CEV model is described. For this purpose, the framework developed under section 3.4 is employed. First, recall the steps introduced in section 3.3 and how the value of a swaption in equation (8) was derived, by choosing the $PVBP$ as the numeraire asset under the forward swap measure $\mathbb{S}$. Then consider the power law model

$$dFS_t = \alpha_t {FS_t}^{\beta} dW_t$$

which is the CEV model written in terms of the forward swap rate FS_t. The implied volatility for this model is according to the approximation formula in equation (9 a)

$$\sigma_t = \frac{a}{f_{av}^{1-\beta}} \left\{ 1 + \frac{1}{24}(1-\beta)(2+\beta)\left(\frac{FS_t - K}{f_{av}}\right)^2 + \frac{1}{24}(1-\beta)^2 \frac{a^2(T_0 - t)}{f_{av}^{2-2\beta}} + \dots \right\}$$

where f_{av} and a are given by equations (9 b) and (9 c). To match the different expiry dates, one needs to change a by a standard numerical procedure, e.g. the Newton-Raphson method. In contrast to more time-consuming methods, as for instance Monte Carlo methods, the implied volatility is obtained from equation (9) and is substituted into the Black valuation formula in equation (8).

It should be noted that equation (9) is not an exact formula, but the error that arises when one uses this approximation for the implied volatility is insignificant, hardly ever approaching 1/1000 of the extrinsic value[24] of the option. The errors arising from the approximation are found to be worst in the case when choosing an elasticity with the resulting $\beta = 0$.[25] This is also to be expected since this case is the extreme point of the power law model in contrast to the log-normal Black model.

4 Risk parameters and hedging strategies

In this part, partial derivatives are introduced. These lay the practical basis for the dynamic hedging techniques developed for later.

4.1 Risk parameters

When one sells a derivative security in the OTC market, there is a problem to manage the risk. If the derivative is the same as one that is traded on the exchange, it is easy to neutralize the risk exposure by buying the same derivative that was sold. When the derivative is not a standardized product on the exchange, the hedging of the risk exposure is not as straightforward. The progress of this thesis will provide a study of a hedging strategy that only aims to neutralize the delta exposure. Consequently, a portfolio that is neutralized in terms of delta exposure is for that reason referred to as a risk-free portfolio. As a result of this limitation, delta is the main risk parameter developed for in this thesis. However, as the delta parameter is dependent on a number of other risk parameters, these partial derivatives are mentioned briefly. As the compounded delta parameter in the CEV model includes a risk parameter vega, a separate section for this parameter is also provided.

4.1.1 Delta

In this section, the delta parameter Δ is introduced, which is the first derivative of the value of a derivative security w.r.t. the underlying security. Delta is an important parameter when hedging swaptions, as it is the sensitivity of a change in the value of the derivative security that relates to the

[24] The extrinsic value or the time value of the swaption is the difference between a swaption's price and its intrinsic value, where the intrinsic value of a swaption is the *in-the-money* proportion of the swaption's premium.
[25] See Hagan, Patrick S. and Diana E. Woodward, 1998, p. 2.

value of the underlying security. If the price of a general derivative security is denoted D and the price of its underlying security is denoted H, the delta parameter is obtained by

$$\Delta = \frac{\partial D}{\partial H} \tag{10}$$

The relationship between the value of a derivative security and its underlying security is illustrated in figure 6, where the slope of the curve is the delta of the derivative.

Value of a call derivative security w.r.t. the underlying security

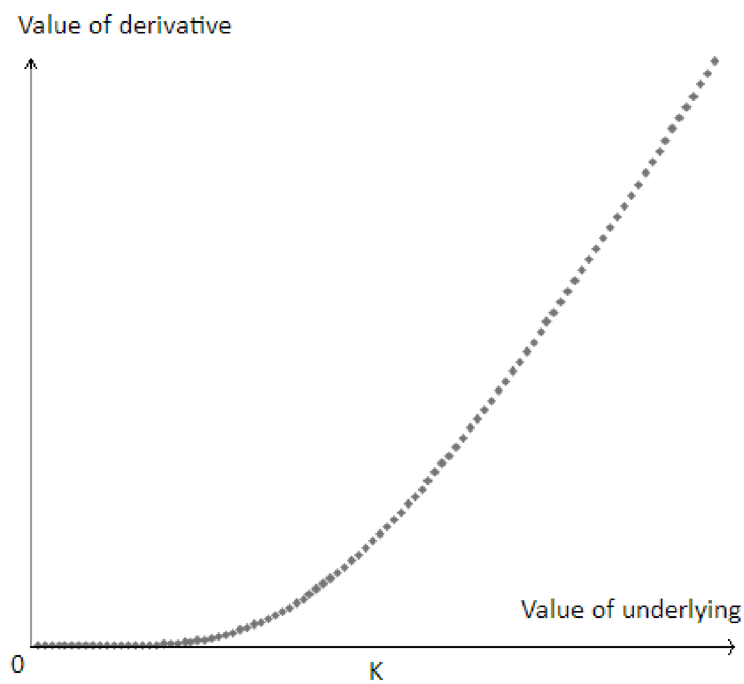

Figure 6: The value of a call derivative security w.r.t. the underlying security. The slope of the curve is the delta of the derivative security and the value varies from zero to one.

As the delta parameter changes instantaneously over time, the calculated delta exposure is valid only for a short period of time. A call option has a delta between zero and one and a put option a delta between minus one and zero. One could think of the absolute value of delta as the probability that the derivative expires *in-the-money* given that the market moves under Brownian motion. Furthermore, the delta of a derivative security varies with gamma, theta and vega. Gamma is the second derivative of the underlying security, i.e. the slope of the delta. The delta variation w.r.t. the underlying security is shown in figure 7.

Variation of delta with the value of the underlying

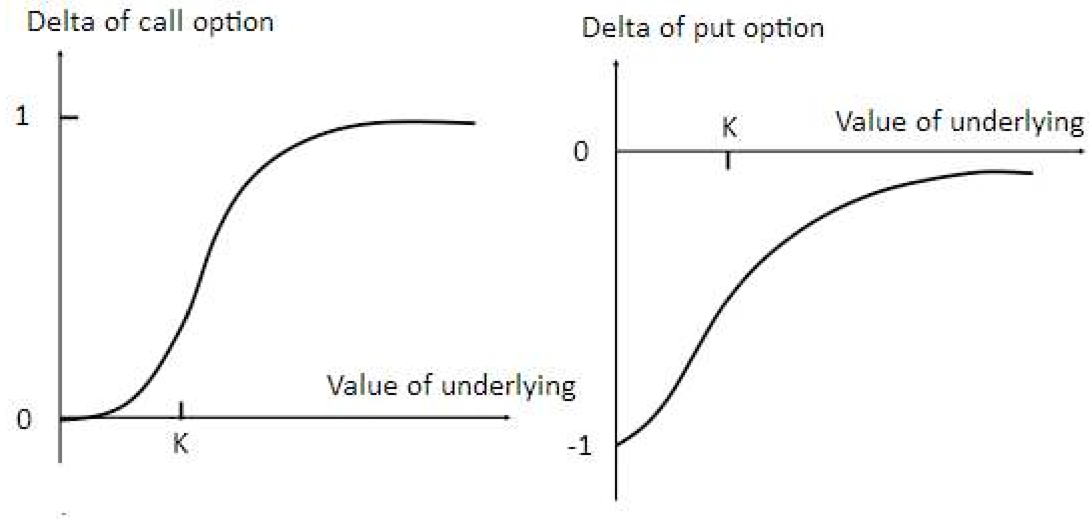

Figure 7: Variation of call option and put option deltas with the value of the underlying.

Furthermore, the delta also varies with time to expiry for the derivative security. This partial derivative is known as the theta of the option. How the delta varies with time to expiry is shown in figure 8.

Variation of delta with time to expiry for a call option

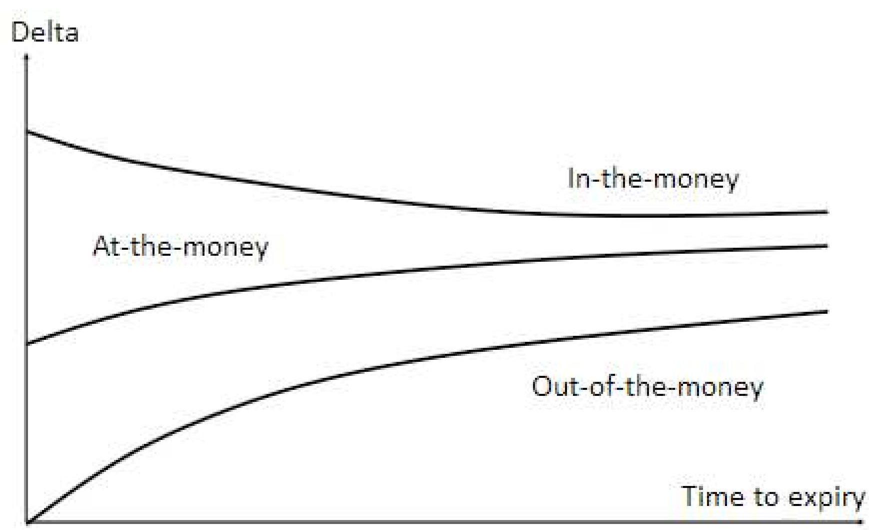

Figure 8: Typical patterns for variation of delta with time to expiry for a call option. The three lines denote *in-the-money*, *at-the- money* and *out-of-the-money*.

Note the pattern in figure 8 that shows that the *in-the-money* delta decreases with time to expiry, while the *out-of-the-money* delta increases with time to expiry.

4.1.2 Vega

In this section, the vega parameter $\mathcal{V}$ is introduced, which is the sensitivity of the derivative security value to the volatility of the underlying security, i.e. the first derivative of the derivative security w.r.t. the volatility. If the value of a general derivative security is denoted D and the volatility of its underlying security is denoted σ_H, the vega parameter is defined as

$$\mathcal{V} = \frac{\partial D}{\partial \sigma_H} \tag{11}$$

If only neutralizing the risk exposure in terms of delta risk, one implicitly assumes that the volatility of the underlying security of the derivative is constant. In practice, volatilities also vary over time. The value of a derivative security is therefore liable to a change in value, due to the movements in the volatility as well as the fluctuations of the value of the underlying security. However as mentioned before, the vega parameter is not included in the Black model approach here employed for hedging swaptions. In this thesis it merely plays a role as an including factor when compounding the delta risk in the CEV model. The variation of vega with stock price for an option is shown in figure 9.

Variation of vega with stock price

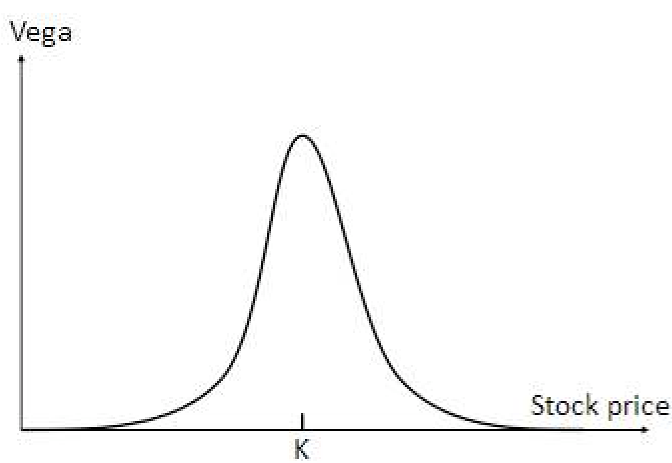

Figure 9: Variation of vega with stock price for an option.

4.2 Risk parameters of the models

This section states the risk parameters for the Black model and the CEV model respectively.

4.2.1 Delta Black model

For the development of the delta in the Black model, recall that the value of a swaption was given by equation (8) and the general derivation of the delta. In the context of swaptions, D is the value of a payer swaption, $P_{P.Swaption;\,t}$, and H is the forward swap rate, FS_t. By analogy to equation (10), the delta of a European payer swaption is

$$\Delta_{P.Swaption;\,t} = \frac{\partial V_{P.Swaption;\,t}}{\partial FS_t} = \alpha \sum_{i=1}^{n} D_{tT_i}\, \Phi(d_1)$$

Equivalently, the delta of a European receiver swaption is

$$\Delta_{R.Swaption;\,t} = \alpha \sum_{i=1}^{n} D_{tT_i}\, (\Phi(d_1) - 1)$$

4.2.2 Delta CEV model

The delta parameter in the CEV model is obtained by the same framework as developed in section 3.4 for resolving the implied volatility for the CEV model, by writing equation (9) as

$$\sigma_t \equiv \sigma_t\big(FS_t;\ K;\ (T_0 - t)\big) \tag{12}$$

and by writing the price of the swaption as

$$V_{P.Swaption;\,CEV;\,t} = V_{P.Swaption;\,t}\Big(FS_t, K, (T_0 - t)\ \sigma_t\big(FS_t;\ K;\ (T_0 - t)\big)\Big) \tag{13}$$

where $V_{P.Swaption;\,t}$ is the swaption value formula stated in equation (8). Differentiating equation (13) w.r.t. the forward swap rate yields

$$\frac{\partial V_{P.Swaption;\,CEV;\,t}}{\partial FS_t} = \frac{\partial V_{P.Swaption;\,t}}{\partial FS_t} + \frac{\partial \sigma_t}{\partial FS_t}\frac{\partial V_{P.Swaption;\,t}}{\partial \sigma_t} \tag{14}$$

From equation (14) it could be noted that the delta risk in the CEV model consists of two terms. The first term is the standard delta risk from the Black model. The second term arises from the systematic change in the volatility caused by changes in the forward swap rate and is proportional to the vega parameter in the Black model.

4.3 Hedging strategies

This section introduces hedging strategies, where a hedged position refers to a position that is neutral in terms of delta risk exposure. First, general concepts of delta hedging strategies are introduced and thereafter the hedging procedure within the Black model framework and the CEV model framework respectively are being initiated.

4.3.1 Delta hedging

When setting up a delta hedge, one typically wants to obtain a position that is both replicating and self-financing. The general idea of creating a portfolio that is delta neutral is that the price changes of the underlying security are compensated by the price changes of the derivative security. The delta neutral portfolio can be constructed by shorting a unit of the derivative security and going long a quantity Δ of the underlying security and invest/borrow the resulting net cash flow from these positions in the risk-free asset under the risk measure specified. Price increases of the underlying are in this way compensated by price drops of the derivative security and vice versa and the property of self-financing is met. By setting the delta of the portfolio to zero, the risks caused by fluctuations of the underlying security should theoretically and almost practically be eliminated. A significant hedging error would imply that the employed model for hedging the derivative security is ineffective. The delta neutral portfolio is set up by

$$\Delta_{Port} = \frac{\partial P_{Port}}{\partial H} = \Delta \cdot \frac{\partial H}{\partial H} - \frac{\partial D}{\partial H} = \Delta \cdot 1 - \Delta = 0$$

where the quantity Δ depends on both the value of the underlying security and the time to expiry for the derivative security, as can be seen in figure 7 and 8. This implies that the amount of the underlying security in the portfolio must be continuously changed in order to maintain a delta neutral portfolio. A portfolio that is rebalanced is called a dynamic hedge. In contrast, a hedge where the positions initially taken never are adjusted is called a static hedge. To rebalance the portfolio continuously is extremely expensive because one has to buy and sell the underlying security continuously in that case.

4.3.2 Delta hedging swaptions

To set up a delta neutral portfolio with swaptions, suppose that one goes short (long) one swaption contract at t. The procedure of delta hedging the short (long) position in the swaption contract, is to go long (short) an amount of the underlying forward swap contract at t and to go long/short an amount equal to the resulting net cash flow from these two positions in the $PVBP$ at t, in order to fulfill the property of a self-financed position. To go long/short the net resulting amount in the $PVBP$ is logical if one recalls that the $PVBP$ is the numeraire asset under the forward swap measure $\mathbb{S}$, when pricing swaptions. The exact amount forward swap contracts needed, w, to achieve a delta neutral portfolio is obtained by setting the delta exposure of the portfolio components that carry delta risk equal to zero. The swaption contract and the forward swap contracts are all dependent on the floating rate, which in turn implies that they carry delta risk. Consequently, w is solved for, by setting the delta net exposure from these positions equal to zero and by rearranging the terms.

$$w\Delta_{P.Swap;\,t} - \Delta_{P.Swaption;\,t} = 0$$

$$\Rightarrow w = \frac{\Delta_{P.Swaption;\,t}}{\Delta_{P.Swap;\,t}}$$

This gives, that the property of a delta neutral portfolio is obtained by going long (short) an amount w in forward swap contracts for every unit short (long) in swaptions. Thereafter, to fulfill the property of a self-financing position, the resulting net cash flow at t, from going short (long) one unit of swaption contract and going long (short) an amount of w forward swaps is computed. The net amount is invested/borrowed in x units of the $PVBP_t$. The parameter x is solved for by setting the net cash flow of the portfolio at t to zero and rearranging the terms.

$$V_{P.Swaption;\,t} - wV_{P.Swap;\,t} - xPVBP_t = 0$$

$$\Rightarrow x = \frac{V_{P.Swaption;\,t} - wV_{P.Swap;\,t}}{PVBP_t}$$

4.3.3 Hedging swaptions with the Black model

Hedging swaptions within the framework of the Black model could be carried out with different methods that are all statically equivalent.[26] The difference between the methods is how they would be employed in practice. The strategy of hedging swaptions in this section closely follows the methodology introduced in section 4.3.2. When setting up the delta neutral hedging strategy in the light of the Black model, recall that the value of a swap and a swaption in the Black model was given by equation (5) and (8). In order to set up the delta neutral hedge, their corresponding deltas, i.e. their first derivative w.r.t. to the forward swap rate, are needed.

$$V_{P.Swap;\,t} = \alpha \sum_{i=1}^{n} D_{tT_i}\,(FS_t - K)$$

$$\Delta_{P.Swap;\,t} = \frac{\partial V_{P.Swap;\,t}}{\partial FS_t} = \alpha \sum_{i=1}^{n} D_{tT_i}$$

$$V_{P.Swaption;\,t} = \alpha \sum_{i=1}^{n} D_{tT_i}\,[FS_t \Phi(d_1) - K\Phi(d_2)]$$

$$\Delta_{P.Swaption;\,t} = \frac{\partial V_{P.Swaption;\,t}}{\partial FS_t} = \alpha \sum_{i=1}^{n} D_{tT_i}\,\Phi(d_1)$$

Setting the delta exposure equal to zero and rearranging the terms yield

$$w\Delta_{P.Swap;\,T_0} - \Delta_{P.Swaption;\,t} = 0$$

$$\Rightarrow w = \frac{\Delta_{P.Swaption;\,t}}{\Delta_{P.Swap;\,t}} = \frac{\alpha \sum_{i=1}^{n} D_{tT_i}\,\Phi(d_1)}{\alpha \sum_{i=1}^{n} D_{tT_i}} = \Phi(d_1)$$

This gives that the property of delta neutrality is obtained by going long (short) an amount $\Phi(d_1)$ in forward swap contracts. Next, to fulfill the property of a self-financing portfolio, the resulting net cash

[26] Equivalent methods are independent of term structure dynamic assumptions since the methods could be transformed into the other methods by a static portfolio. For further details on the differences of the models, see Barton, Geoff, Tim Dun and Erik Schlögl, 2001, p. 5.

flow at t from going short (long) one unit of swaption contracts and going long (short) an amount of $\Phi(d_1)$ forward swaps is computed. The net amount is invested/borrowed in $PVBP_t$.The amount invested/borrowed in the $PVBP_t$ is solved for by setting the net cash flow of the portfolio at t to zero and by rearranging the terms.

$$V_{P.Swaption;\,t} - \Phi(d_1)V_{P.Swap;\,t} - xPVBP_t = 0$$

$$\Rightarrow x = \frac{V_{P.Swaption;\,t} - \Phi(d_1)V_{P.Swap;\,t}}{PVBP_t}$$

$$\Rightarrow x = \frac{\alpha\sum_{i=1}^{n} D_{tT_i}\,[FS_t\Phi(d_1) - K\Phi(d_2)] - \Phi(d_1)\alpha\sum_{i=1}^{n} D_{tT_i}\,(FS_t - K)}{\alpha\sum_{i=1}^{n} D_{tT_i}} = K[\Phi(d_1) - \Phi(d_2)]$$

At T_0, the value of the portfolio is examined. If the resulting error from the swaption hedge largely deviates from zero, this would imply that hedging swaption contracts with the Black model could be questioned. Thus, a significant error would suggest that more complex models should be employed when hedging these derivative securities.

4.3.4 The Dun et al. approach

This section describes the underlying swap method.[27] The underlying swap method closely follows the general approach introduced above, but has an elegant way of dealing with the weight assigned to the $PVBP$ when employing the Black model. In this scenario, assume that the swaption is shorted. Recall the swaption pricing formula from equation (10). This could be rewritten as

$$\begin{aligned}
V_{P.Swaption;\,t} &= \alpha\sum_{i=1}^{n} D_{tT_i}\big(FS_t\Phi(d_1) - K\Phi(d_2)\big) \\
&= \alpha\sum_{i=1}^{n} D_{tT_i}\,(FS_t - K)\Phi(d_1) - \alpha K\sum_{i=1}^{n} D_{tT_i}\,[\Phi(d_2) - \Phi(d_1)] \\
&= \Phi(d_1)V_{P.Swap;\,t} - K\big(\Phi(d_2) - \Phi(d_1)\big)\alpha\sum_{i=1}^{n} D_{tT_i} \\
&= \Phi(d_1)\left(V_{P.Swap;\,t} + K\alpha\sum_{i=1}^{n} D_{tT_i}\right) - \Phi(d_2)K\alpha\sum_{i=1}^{n} D_{tT_i} \qquad (15)
\end{aligned}$$

[27] See Barton, Geoff, Tim Dun and Erik Schlögl, 2001, p. 5.

Equation (15) states the swaption price as a sum of the underlying swap and the $PVBP$. Through the assumption of log-normality on the proportion

$$\frac{V_{P.Swap;\,t} + K \cdot PVBP_t}{PVBP_t} = FS_t$$

these quantities become the hedging instruments and the hedging weights can be read directly from equation (15). That is, a replicating hedge to the shorted swaption can be obtained by going long $\Delta = \Phi(d_1)$ units of the underlying swap and short $K[\Phi(d_2) - \Phi(d_1)]$ units of the $PVBP$.

4.3.5 Hedging swaptions with the CEV model

The construction of a hedged position within the framework of the CEV model is very similar to the procedure already developed for, as the CEV model also works with the $PVBP$ as the numeraire asset under the forward swap measure $\mathbb{S}$. Recall, that the CEV model in fact is a generalization of the Black model. The main difference is the resolving of the delta. The procedure of resolving the CEV delta is built on the framework stated in section 3.4. First, recall the derivation of the delta from section 4.2.2. Equation (14) states that the CEV delta is

$$\frac{\partial V_{P.Swaption;\,CEV;\,t}}{\partial FS_t} = \frac{\partial V_{P.Swaption;\,t}}{\partial FS_t} + \frac{\partial \sigma_t}{\partial FS_t}\frac{\partial V_{P.Swaption;\,t}}{\partial \sigma_t}$$

This equation states that the CEV delta comprises the Black model delta and a term proportional to the Black vega. The vega parameter is obtained by analogy to equation (11) as

$$\mathcal{V}_t = \frac{\partial V_{P.Swaption;\,t}}{\partial \sigma_t} = \frac{\partial \alpha \sum_{i=1}^{n} D_{tT_i}\left(FS_t\Phi(d_1) - K\Phi(d_2)\right)}{\partial \sigma_t}$$

$$= \alpha \sum_{i=1}^{n} D_{tT_i} FS_t\sqrt{(T-t)}\varphi(d_1)$$

The other factor, proportional to the Black vega, is computed as the first derivative of equation (12) w.r.t. the forward swap rate and is given by

$$\frac{\partial \sigma_t}{\partial FS_t} = \frac{\partial\left(\frac{a}{f_{av}^{1-\beta}}\left\{1 + \frac{1}{24}(1-\beta)(2+\beta)\left(\frac{FS_t - K}{f_{av}}\right)^2 + \frac{1}{24}(1-\beta)^2\frac{a^2(T_0-t)}{f_{av}^{2-2\beta}} + \ \dots\right\}\right)}{\partial FS_t}$$

$$= \frac{a(\beta - 1)}{2} f_{av}^{\beta-2} + \frac{a(1-\beta)(2+\beta)}{24}\left(2(FS_t - K)f_{av}^{\beta-3} + \frac{(FS_t - K)^2(\beta - 3)f_{av}^{\beta-4}}{2}\right)$$
$$+ \frac{(1-\beta)^2}{48} a^3 (T_0 - t)(3\beta - 3) f_{av}^{(3\beta-4)}$$

Once the CEV delta is resolved, the method of setting up a hedged position follows the Black model hedging procedure, introduced in section 4.3.3.

5. Methodology

In this part, the step-by-step methodology for evaluating the models in practice, in terms of swaption hedging, is presented.

5.1 Dynamic hedging

In order to evaluate the effectiveness of hedging derivatives with a specific model empirically and in order to compare the preciseness of models, a dynamic hedging portfolio can be set up. Thereafter, potential hedging errors of each portfolio can be estimated and the portfolios can be rebalanced in terms of delta exposure at discrete time intervals, Δt, i.e. new deltas must be computed for every rebalance date along the date structure specified. Since the delta risk in reality changes instantaneously over time w.r.t. the underlying security, dynamic hedges are necessarily only approximations of the continuous, replicating and self-financing strategies specified by the models. Therefore, either the self-financing or the replicating property is lost to some extent.[28]

5.2 Evaluating the models

This study chooses to perform the portfolio strategy in a self-financing manner, which implies that the hedges will replicate only on average. The approximation should nevertheless make any conclusions potentially drawn for hedging strategies in practice spurious since such models necessarily also must be discrete. Moreover, the potential error that arises from the approximation could be diminished by a high rebalancing frequency within the hedging strategies. Before the step-by-step methodology is developed, the date structure introduced in section 2.3 is expanded to the following discrete date structure

[28] See Barton, Geoff, Tim Dun and Erik Schlögl, 2001, p. 2.

$$t_0 < t_1 < t_2 \dots t_{n-1} < t_n = T_0 < T_1 < T_2 \dots T_{n-1} < T_n$$

In order to perform the dynamic hedge, weights are calculated for every t along the date structure. First, the weights of the forward swap and $PVBP$, needed at t_0 to fulfill the properties of a replicating and self-financing portfolio, are computed. When the hedging procedure for the Black model is carried out the Dun et al. approach is followed.

At t_0, Short one unit of a swaption contract, $V_{P.Swaption;\, t_0}$

Long $\Delta = \Phi(d_1)$ units of the underlying swap, $V_{P.Swap;\, t_0}$

Short $K[\Phi(d_2) - \Phi(d_1)]$ units of the $PVBP_{t_0}$

By analogy to equation (15), the net cash flow from this strategy is equal to zero as

$$V_{P.Swaption;\, t_0} = \Phi(d_1)V_{P.Swap;\, t_0} - K\big(\Phi(d_2) - \Phi(d_1)\big)\alpha \sum_{i=1}^{n} D_{t_0 T_i}$$

$$\Rightarrow V_{P.Swaption;\, t_0} - \Phi(d_1)V_{P.Swap;\, t_0} + K\big(\Phi(d_2) - \Phi(d_1)\big)\alpha \sum_{i=1}^{n} D_{t_0 T_i} = 0$$

At t_1, The value of the portfolio set up at t_0 is computed. The portfolio is rebalanced by going long (short) an amount of forward swaps in order to maintain a delta neutral position with the swaption contract. Subsequently, the portfolio has a new net cash flow from the swaption and the forward swap positions taken and this amount is borrowed (invested) in $PVBP$ from t_1 until the next observation date t_2.

At t_2, The value of the portfolio set up at t_1 is computed. Again, the portfolio is delta neutralized by assigning new weights to the hedging instruments for the next time interval.

This procedure is carried out at every defined date on the defined date structure until the swaption contract expires at T_0. The hedging procedure for the CEV model generally follows the methodology introduced. However, instead of assigning the positions' weights suggested by the Dun et al. approach, the general approach of hedging swaptions is followed.

The computed hedging errors are sorted on swaption contracts to be able to compare the selection criteria within the Black model and to compare the two different diffusion class models with each other, which are the two main purposes of this thesis. To also analyze the impact of the strike offset for the hedging performance, the hedging errors are bundled in three intervals w.r.t. the difference between the forward swap rate and the strike rate, i.e. the strike offset. To detect any potential impact of the hedging performance w.r.t. time to expiry, the errors are also sorted on their hedging period.

To test whether the Black model can be improved by changing the selection criterion for the implied volatility, three different selection criteria for the volatility are tested for. These are to be denoted, strike volatility (*K-Vol*), at-the-money volatility (*ATM-Vol*) and average volatility (*AVG-Vol*). In the *K-Vol* scenario, the selected volatility is the one that corresponds to the implied market volatility for the swaption hedged. When hedging one swaption contract, this way of selecting the volatility is appropriate but when hedging swaptions with different strikes and time to expiry, this essentially means that different models are being used for each strike and time to expiry. In the *ATM-Vol* scenario, the selected volatility will always be the one that corresponds to an *at-the-money* swaption at t_i. When managing large books of swaptions, this could be a natural selection of volatility if one not wants to violate the crucial assumption of constant volatility, as this swaption generally is the most liquid swaption and consequently has a volatility closer to the fair volatility. The *AVG-Vol* scenario is the constructed selection criterion that is specified in this thesis in order to investigate the first research question. This selected volatility is the average volatility of nine swaptions trading with a strike offset spanning from *-2%* to *+2%* of an *ATM* swaption w.r.t. the forward swap rate at t_i. Seven of these implied volatilities are between *-1%* and *+1%* of an *ATM* swaption.

To test for the second research question, the CEV model is tested for two different elasticity values in addition to the special case of $\psi = 2$, which is the Black model. The CEV model elasticity cases tested for is $\psi = 1$ and $\psi = 0$, referred to as the $\beta = 0{,}5$ and $\beta = 0$ scenarios in line with the terminology stated in section 3.4.1. When applying the CEV model for the dynamic hedge, the algorithm to calculate an approximation for a in Appendix B is used. The algorithm is employed instead of numerical procedures, e.g. the Newton-Raphson method, as suggested in section 3.4.2.

6. Data

In this part, the description and preparation of the data used for the hedging study is presented.

6.1 Data description

The data sample used for the analysis was provided by the Department of Finance at Stockholm School of Economics.[29] The data is obtained from the European market during the period July 31, 2002 to October 17, 2007 and offers thirteen observations per year. In practice, a limit is defined for the delta exposure, which is often expressed as the equivalent maximum position in the underlying forward swap. Rebalancing of the portfolio is carried out when the delta rises over this limit.[30] However, the data does not allow for the dynamic hedge to follow such procedure, why the hedging study will be conducted with fixed time intervals of 1/13 year, i.e. 28 days, between every rebalancing date.

The choice of data is interesting in many aspects. It covers both a period of financial downturn in the beginning of the sample period and a longer period of financial upturn, which eventually landed in a financial boom. The datasheet embraces both periods of relatively calm market conditions and periods of financial movements, which in turn affects the volatility of the interest rates.

The data sample consists of one year Forward *LIBORs* and market volatilities for swaption contracts on five year swap contracts with strikes spanning from -2% to +2% relatively to the prevailing forward swap rate. The data comprises yearly swaption volatilities, with the first data points five year prior expiry.

When setting up the dynamic hedge, the hedged contract is a swaption that is *ATM* three years prior expiry. This provides 30 unique swaptions that could run to expiry. To avoid statistical flaws, due to a small sample size, and to acquire more data points, the data set is extended to also include swaption contracts that trade at *+1%* and *-1%* respect to an *ATM* swaption three years prior expiry.

6.2 Data preparation

To match the one year forward *LIBORs* to the observation dates, a fitting term structure for the interest rates is linearly interpolated between the forward *LIBOR* quotes with expiry dates ranging from one year to seven years, in steps of 28 days. The spot rate is estimated as a linear extrapolation between the

[29] Linus Kaisajuntti, Department of Finance at Stockholm School of Economics.
[30] See Hull, John C., 2006, p. 363.

forward *LIBORs* with expiry dates ranging from one to two years. The implied market volatilities are also linearly inter- and extrapolated to steps of 28 days, in order to provide volatilities for all relevant expiry dates at each rebalancing date. Furthermore, the implied volatility used in the models is collected from the original data sheet by using a linear interpolation between the volatilities for the given strike offsets.

7. Results

This part provides the results from the hedging study. First, the results from the tests on the hedging characteristics for the Black hedging approach are presented and the impact that the strike offset and the time to expiry have for the performance of the Black model is inspected. Thereafter, the two research questions are treated and finally a discussion of the difference between the Black and the CEV approach is provided. First, one should be aware that when hedging swaptions in practice, there will always be hedging errors even if all other model assumptions are fulfilled, as the rebalancing of the portfolio is carried out at discrete time intervals. Second, the relative importance of each contributing factor for the hedging error is hard to distinguish in this thesis since the exogenous factors strike offset, time to expiry and the volatility jointly explains the hedging error. However, this should not have large influences on the main purpose of this thesis, i.e. to compare different selection criteria and models of different diffusion classes. The comparisons potentially include more noise than a hedging study that could separate each explaining factors' individual contribution, but the comparison of the models should however not be invalidated as this thesis only strives to make a relative comparison of the hedging performance of the models.

7.1 Hedging characteristics

This section examines the hedging performance of the Black model w.r.t. the strike offset and the time to expiry. Thereafter, a brief discussion of the obtained results will follow in the light of the hedging characteristics one would expect to realize according to theory.

7.1.1 The impact of strike offset

The first test examines the impact of the strike offset for the hedging performance. The results from this test are shown in figure 10.

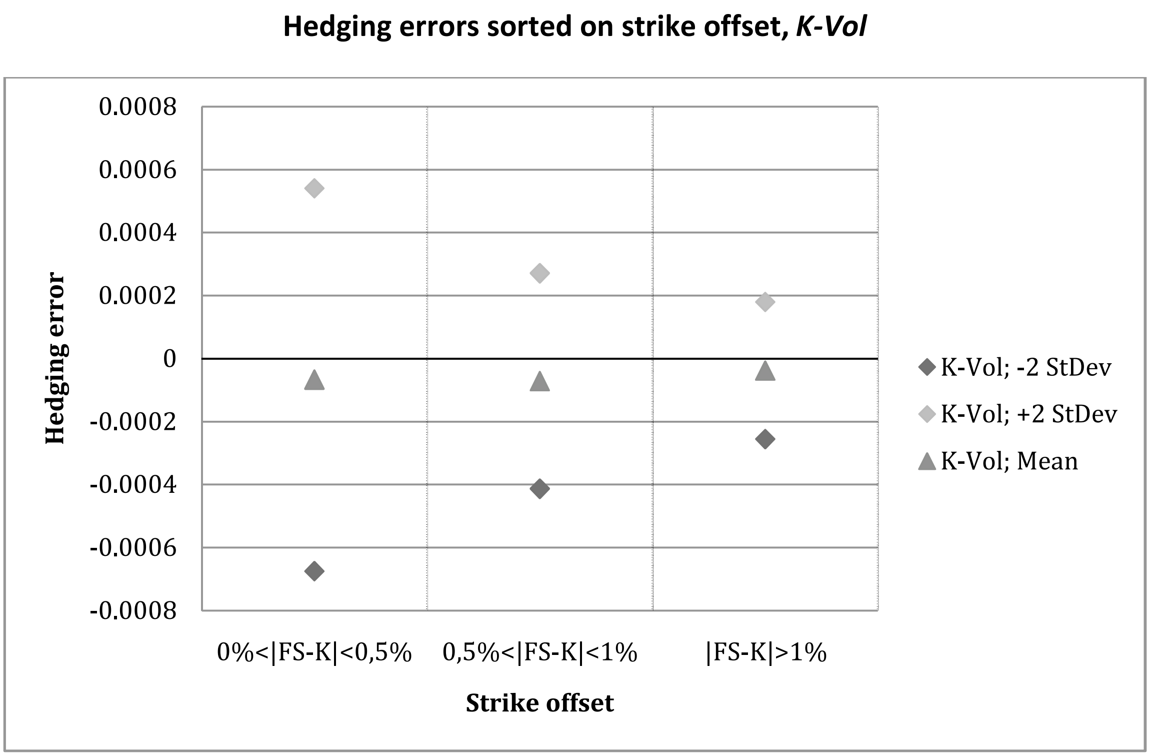

Figure 10: The interval between -2/+2 standard deviations and the mean of the hedging errors are plotted for each strike offset, i.e. the difference between the forward swap rate and the strike rate. The swaptions with strike rates that start *ATM*, *+1%* and *-1%* in the first hedging period are bunched together.

Figure 10 shows a trend in the hedging errors, indicating that the hedging errors are greater when the forward swap rate is closer to the strike rate and smaller when they are further apart, i.e. the hedging error is a decreasing function of the absolute strike offset.

7.1.2 The impact of time to expiry

The second test attempts to examine whether any trend could be detected for in the hedging errors w.r.t. time to expiry of a swaption. The results from this test are shown in figure 11.

Hedging errors sorted on hedging period, *K-Vol ATM*

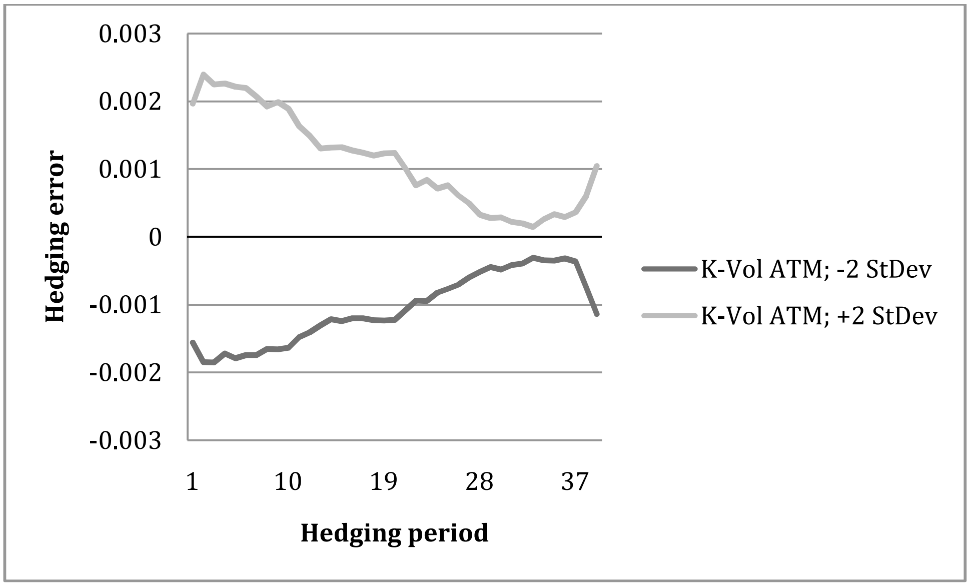

Figure 11: The two lines represent the interval between -2/+2 standard deviations of the hedging errors for each hedging period, where the first period starts three years prior expiry and the last period starts one month prior expiry. The strike rate of the swaptions starts *ATM* in the first hedging period.

The trend of the hedging errors is downward sloping, indicating that it is easier to hedge closer to expiry of the swaption. As can be seen in Appendix A in figure A1 and A2, this trend holds for all different swaptions hedged, i.e. swaptions with strikes beginning *ATM*, *-1%* and *+1%*, although the trend is much less palpable for the swaption that starts at *-1%*.

7.1.3 Discussion of the hedging characteristics

In order to compare the results from the Black model hedging study with what theory suggests, the payoff of a payer swaption in figure 4 is plotted in figure 12 together with the value of a swaption w.r.t. the forward swap rate from figure 6.

Swaption value w.r.t. the forward swap rate versus the payoff of payer swaption

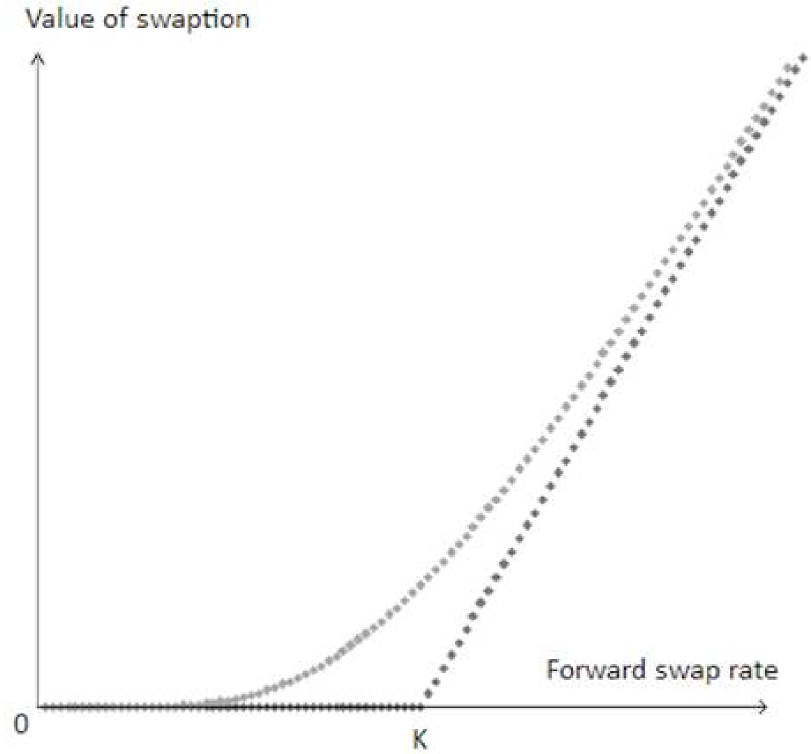

Figure 12: The convex line plots the swaption value w.r.t. the forward swap rate and the straight line plots the payoff of a payer swaption.

As can be seen in figure 12, the error between the two curves should, on average, be largest when hedging a swaption contract that has a delta, the slope of the convex curve, close to 0,5. Thus, theory suggests that the error arising from hedging a swaption contract depends on the uncertainty whether the swaption will expire *in-the-money* or not. When the forward swap rate has drifted away from the strike rate, the swaption value is almost identical to the payoff, which indicates that the hedging error is low. This is because $\Phi(d_1)$, the probability that the swaption expires *in-the-money*, is close to zero or one. This means that there is almost no uncertainty of what will happen at expiry of the swaption. When the forward swap rate is close to the strike rate, $\Phi(d_1)$ is closer to 0,5, i.e. the uncertainty of what will happen at expiry is high due to a large exposure to changes in the forward swap rate. Hence, the average hedging error should be higher. Figure 10 plots the errors obtained in the hedging study when bundling the errors w.r.t. their strike offset. The result emphasizes what theory suggests, that it becomes relatively more difficult to hedge when the forward swap rate is close to the predetermined strike rate. In section 4.1.1, the option delta was also stated as a function of time to expiry in figure 8. The impact of time to expiry for the swaption value versus the payoff function of a payer swaption is plotted in figure 13.

Swaption value for different expiry dates w.r.t. the forward swap rate versus the payoff of a payer swaption

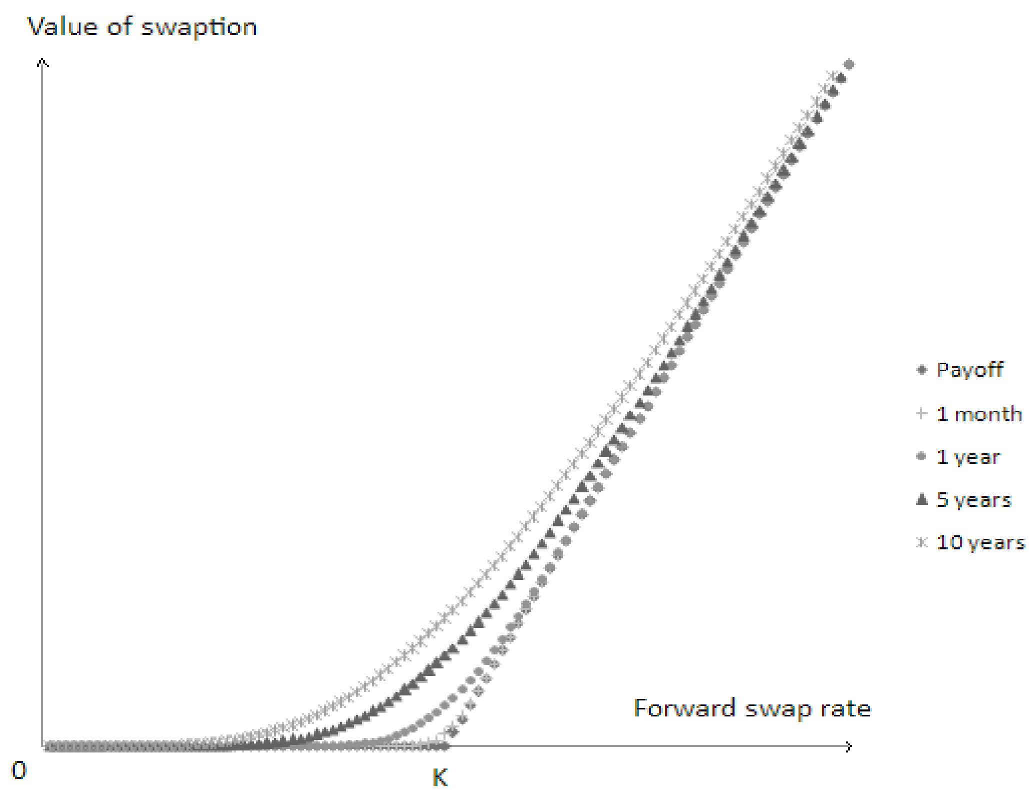

Figure 13: The convex lines plot the swaption value for different expiry dates w.r.t. the forward swap rate and the straight line plots the payoff of a payer swaption.

Figure 13 shows that the value of a payer swaption shifts outward as the time to expiry increases. This figure also shows, by analogy to figure 8, that the delta for an *out-of-the-money* payer swaption increases with more time left to expiry, while the delta for an *in-the-money* payer swaption decreases with more time left to expiry. The changes of the delta of an *in-the-money* or an *out-of-the*-money payer swaption is relatively small compared to the differences over the same period in the deltas of swaptions that is deep *in-the-money* or deep *out-of-the-money*. The errors from the hedging study, when sorting on hedging periods in figure 11, support what theory suggests. However, the impact that the strike offset has on the hedging errors, when sorted on hedging periods, is hard to distinguish from the impact of time to expiry. In figure A2, where the errors w.r.t. the time to expiry are reinforced by the strike offset, the trend is clear. That time to expiry has some explanatory power for the hedging performance of the Black model should however be clear as both figure 11 and A1, where the strike offset is closer to zero, indicate a positive relationship between the hedging errors and time to expiry.

Theory suggests that the volatility affects the hedging performance in the same manner as the time to expiry, i.e. has a relatively higher importance for swaptions traded deep *in-* or *out-of-the-money*, implying an outward shift of the convex curve in figure 12 for higher volatilities. This relationship can be seen in Appendix A in figure A3. For this specific effect, no explicit tests were carried out.

7.2 Selection criteria for the volatility in the Black model

The first research question deals with the Black model and whether the performance can be improved by changing the selection criterion for the volatility. In order to get a better picture of the performance of the Black model under each of the volatility selection criteria, the volatility structures are presented in figure 14.

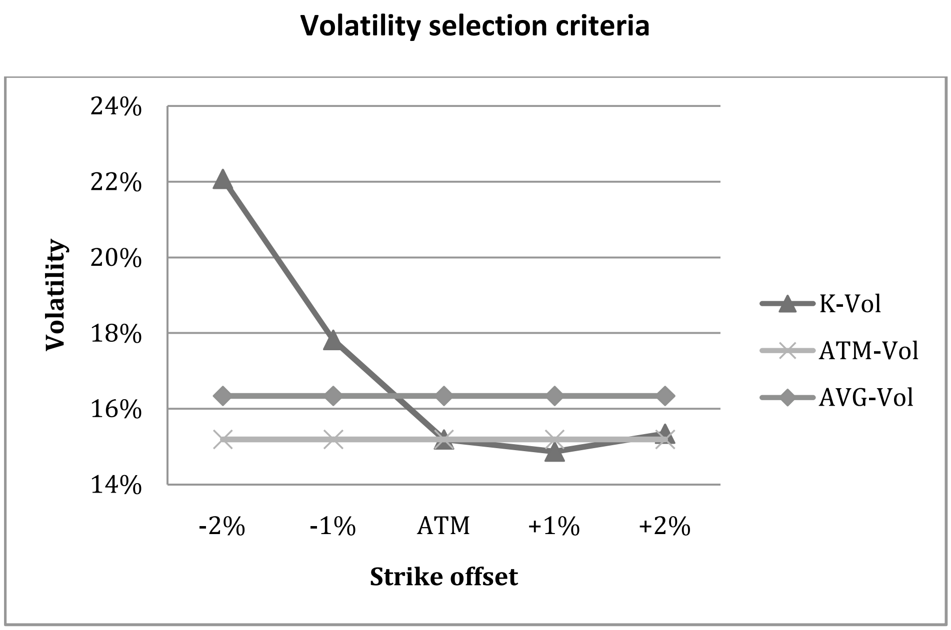

Figure 14: The mean of the implied volatilities in the first hedging period w.r.t. the strike offset for each of the three different selection criteria, the *K-Vol*, the *ATM-Vol* and the *AVG-Vol* scenarios are plotted.

As can be seen, the assumption of constant volatility over different strikes is inconsistent with the actual volatilities observed in the market, which in figure 14 is equivalent to the *K-Vol* scenario. Hence, regardless of choosing the *ATM*, *AVG* or any other σ_t, the Black model is unable to replicate market volatilities because a constant σ_t implies a flat volatility structure. When hedging one single swaption, it is always possible to take the market volatility for each swaption's particular strike and time to expiry for input in the model. The difficulties with the assumption of constant volatility occur when managing large books of swaptions, as the risk calculated at a given strike is not consistent with the same risk calculated at other strikes, which put in uncertainty into the hedging of risks across strikes. In practice though, one chooses different volatilities for each strike and time to expiry, but this basically means that different models are being used.

To analyze the impact of the selection criteria for the hedging performance of the Black model, the resulting hedging errors in each selection criterion is plotted in figure 15.

Hedging errors sorted on swaption contract
The Black model scenarios

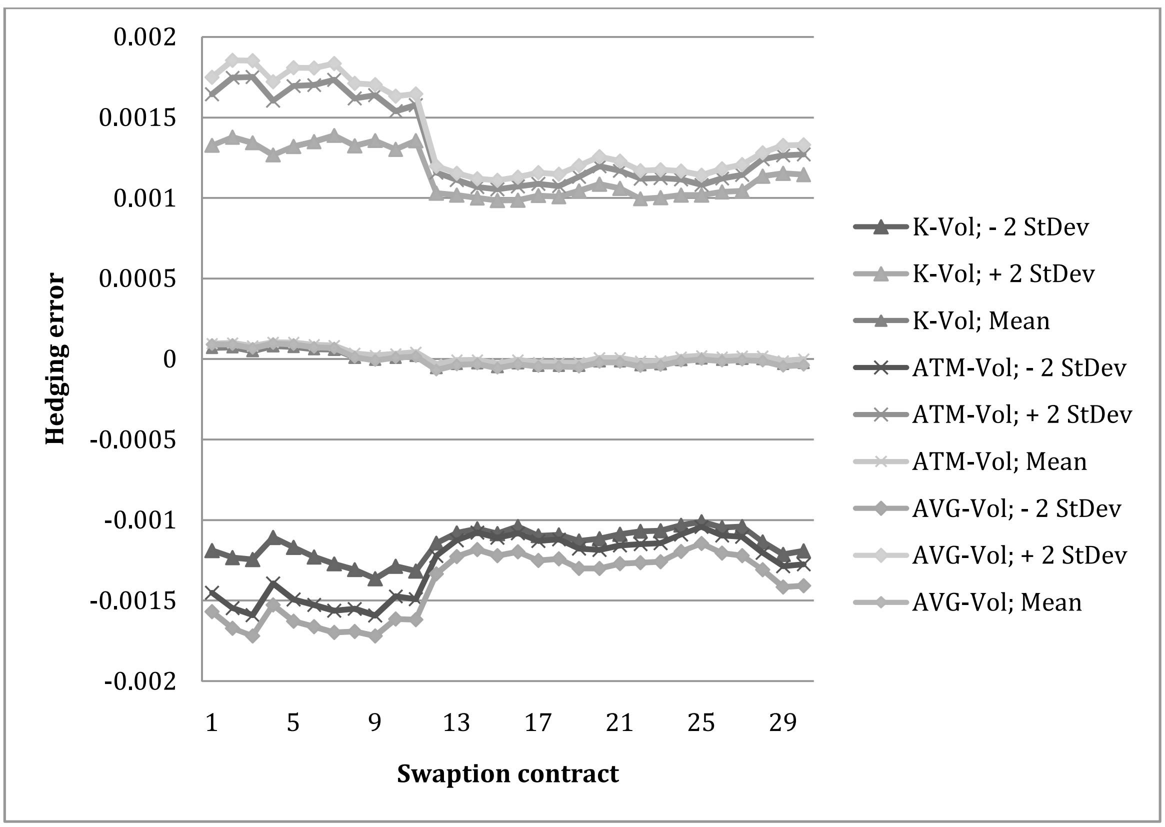

Figure 15: The interval between -2/+2 standard deviations and the mean of the hedging errors for each of the three different volatility scenarios in the Black model are plotted. The swaptions with strike rates that start *ATM*, *+1%* and *-1%* in the first hedging period are bunched together.

It is clear that the *K-Vol* selection criterion improves the result and outperforms the other two chosen volatilities in the Black model. It should however be stressed that this scenario is a strict violation of the underlying assumption of the Black model. The result further shows that the hedging performance in the *ATM-Vol* scenario is superior to the *AVG-Vol* scenario. This indicates that the constructed selection criterion, the *AVG-Vol* scenario, which not violates the underlying Black model assumption, is a worse performer relative to the standard selection criterion, the *ATM-Vol* scenario. The same result is valid when the hedging errors are bundled w.r.t. strike offset in Appendix A in figure A4, although the pattern is not as obvious as in figure 15. One could expect that the hedging performance in the *K-Vol* scenario would be superior already before setting up the test, as this scenario chooses the actual market volatility as the input volatility. However, the market for most swaptions is fairly illiquid, which makes it hard to get a reliably implied volatility of the underlying swap contract. In general, the market for swaptions traded *ATM* is more liquid than the market volatilities for swaptions traded with a strike offset, which implies that the *ATM* market volatility should be closer to the fair market volatility. For this reason, it should be of relevance to test for the hedging performance with different volatility scenarios. The results from the test prove that the impact of the liquidity effect does not outweigh the shortcoming of choosing a flat volatility structure.

7.3 The Black model versus the CEV model

As previously stated, a drawback of the Black model arises when choosing the input value for the implied volatility. The second research question investigates whether the hedging performance can be improved with a model that has a more accurate volatility structure compared to the flat volatility structure assumed in the Black model. When investigating this research question, the *ATM* selection criterion is chosen, as this is the best performer of the scenarios that do not violate the assumption of constant volatility across different strikes and expiry dates. In figure 16, the hedging performance for each swaption contract is compared by plotting the hedging errors for the Black model, i.e. CEV $\beta = 1$, in comparison to the CEV model when $\beta = 0$ and $\beta = 0{,}5$.

Hedging errors sorted on swaption contract
The CEV model scenarios

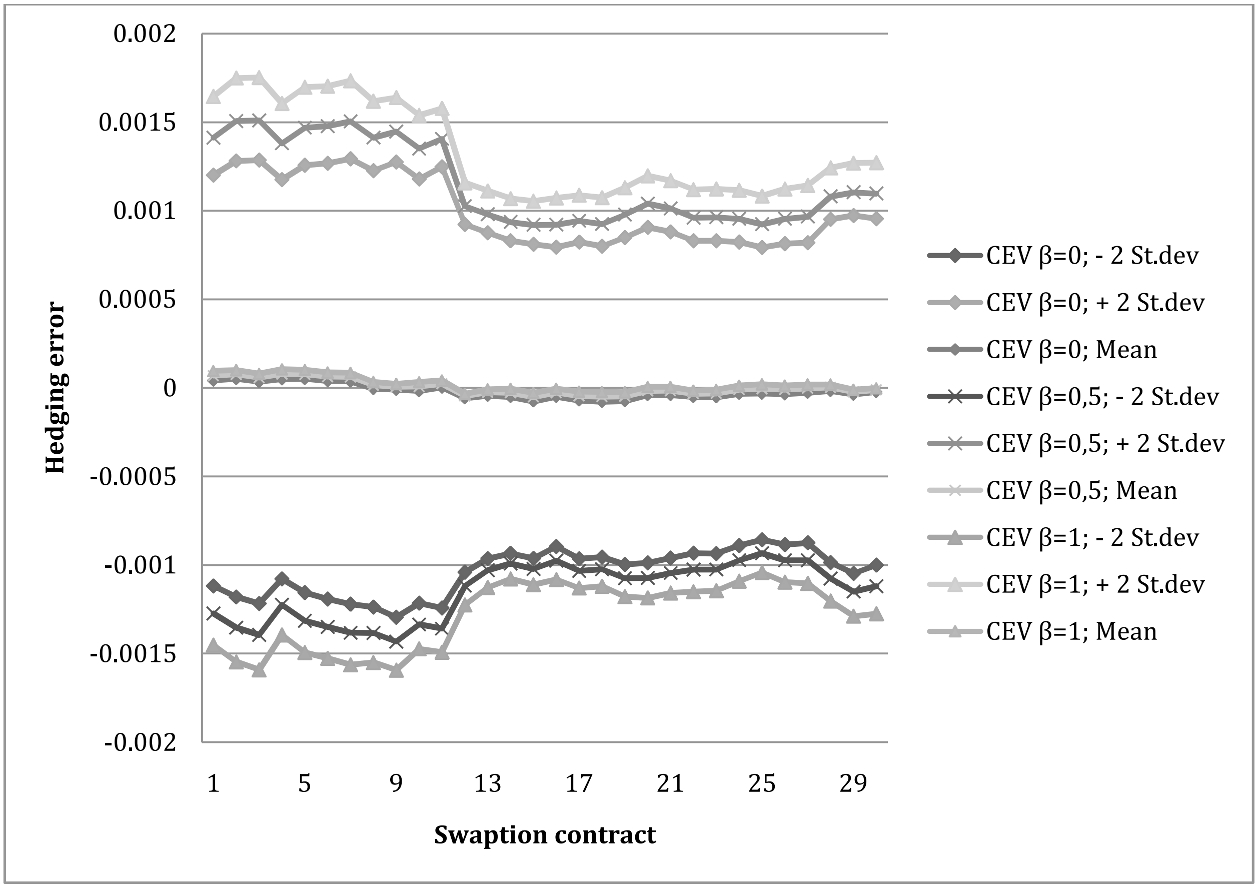

Figure 16: The interval between -2/+2 standard deviations and the mean of the hedging errors for each of the three different β scenarios in the CEV model are plotted. The swaptions with strike rates that start *ATM*, *+1%* and *-1%* in the first hedging period are bunched together.

The results of the hedging performance indicate that the CEV model clearly outperforms the Black model for both $\beta = 0$ and $\beta = 0{,}5$. Within the CEV model, the best hedging performance is achieved when choosing $\beta = 0$. The same result is valid when the hedging errors are bundled w.r.t. strike offset in Appendix A in figure A5, although the pattern is not as obvious as in figure 16.

7.4 A rough guide to why the CEV model outperforms the Black model

The main assumption of the Black model is that the forward swap rate at expiry date of the swaption is log-normally distributed with constant volatility, under the forward swap measure. However, by studying market implied volatilities for swaptions w.r.t. the forward swap rates, one can see that implied volatilities are not constant. This relationship is plotted in figure 17.

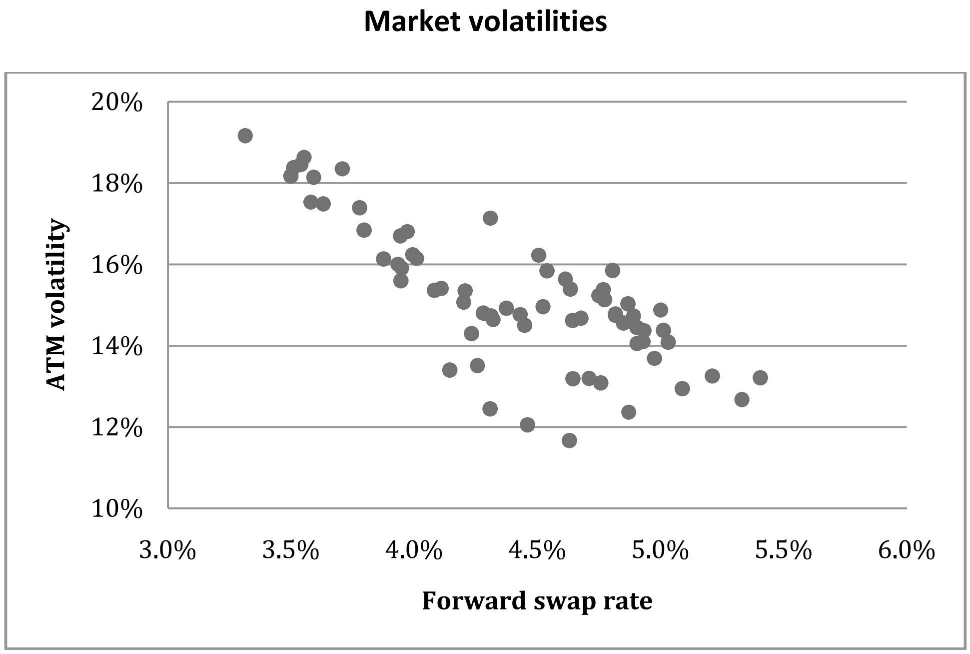

Figure 17: Market data for 3 year swaptions with ATM volatility plotted w.r.t. the forward swap rate. These volatilities are the same as in the *K-Vol* scenario in the Black model.

The results in figure 17, clearly contradicts the main assumption of the Black model. Hence, it is paradoxical that when the market uses the Black pricing framework for quoting market implied volatilities, the results is inconsistent with the Black model. This suggests that models with alternative diffusion classes, that could hedge swaptions at different strikes and expiry dates more properly, should be employed. This thesis has for this purpose focused on the CEV diffusion class. The diffusion class of the CEV model states that the volatility is a direct inverse function of the forward swap rate, which is not the case in the Black model. Economic rationale suggests that an option pricing formula based on CEV diffusion can fit market prices more properly than one based on the Black model, as suggested by figure 5 in section 3.4.1. In the light of the previous discussion, the results from figure 16 can be analyzed. From this figure, it can be seen that the hedging performance of the CEV model clearly outperforms the Black model, which supports that the hedging performance can be improved by choosing a model with a more accurate volatility structure.

Furthermore, the CEV model differs to the Black model in the way the delta risk of the swaption is computed. In the CEV model, the computed delta risk is comprised of the Black delta and the term $\frac{\partial \sigma_t}{\partial FS_t}$, that is proportional to the Black vega, which arises from the systematic change in σ_t caused by changes in the forward swap rate. To study the impact that the additional risk factor has for the hedging performance, a comparison between the hedging performances with different β and the *K-Vol* scenario is carried out. However, this means that the underlying assumption of constant volatility in the Black model is violated. When focusing on the *K-Vol* scenario, where the volatility structure already is fitted to market volatilities, this comparison should indicate the importance of the extra risk factor added. The results of this test are shown in figure 18.

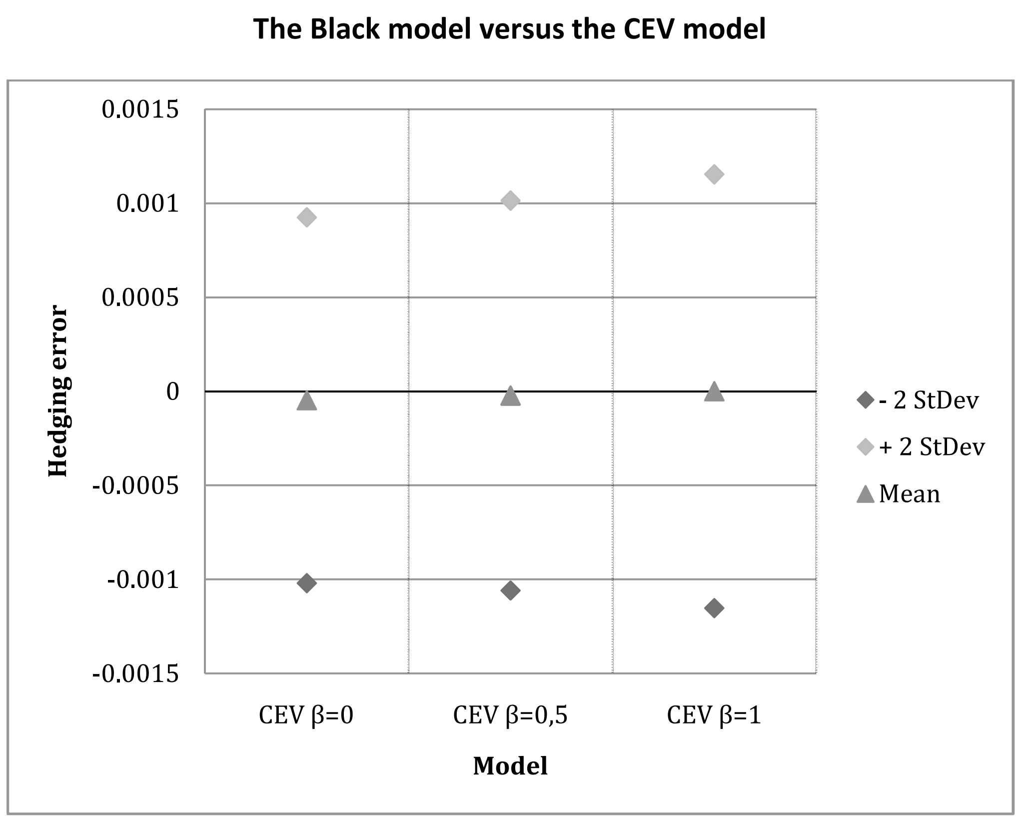

Figure 18: The interval between -2/+2 standard deviations and the mean of the total hedging errors for each of the three different volatility scenarios in the CEV model are plotted. The swaptions with strike rates that start *ATM*, *+1%* and *-1%* in the first hedging period are bunched together.

From figure 18, it can be seen that the hedging errors are lower when $\beta < 1$, compared to the special case of $\beta = 1$, i.e. the *K-Vol* scenario in the Black model. This indicates that the CEV model has something extra to offer. In order to correctly draw any conclusions from the comparison of the two models, one should bear in mind that the vega risk also could be hedged within the Black model by setting up a delta-vega neutral portfolio. This is not examined in this thesis but in general, any vega risk arising from the systematic volatility change w.r.t. the forward swap rate, could be hedged both more properly and inexpensively as delta risk in the CEV model.

8. Conclusion

This thesis studies the impact that strike offset and time to expiry have for the hedging performance. The results emphasize what theory suggests, that it becomes relatively more difficult to hedge when the forward swap rate is close to the predetermined strike rate. When the forward swap rate has drifted away from the strike rate, the hedging error is low. This is because the probability that the swaption expires *in-the-money* is close to zero or one. This means that there is almost no uncertainty of what will happen at expiry of the swaption. When the forward swap rate is close to the strike rate, the uncertainty of what will happen at expiry is high due to a large exposure to changes in the forward swap rate and therefore the average hedging errors are higher. The results of the hedging performance indicate that there is a relationship between the hedging performance and time to expiry. However, from the test performed, it is hard to distinguish time to expiry from the other explanatory variables, in terms of the impact that the variables have on the hedging performance. That time to expiry has some explanatory power for the hedging performance of the Black model should however be clear since all tests indicate a positive relationship between the hedging errors and time to expiry. The conclusion is that the worst scenario for a trader is when the difference between the forward swap rate and the strike rate is small close to expiry date.

In order to examine the research question whether the hedging performance can be improved by changing the selection criterion for the volatility, this thesis compares the performance of the Black model with three different selection criteria for the volatility. The hedging study does not give any support to the research question, as the different volatility selection criteria do not improve the hedging performance, unless the underlying assumption of the Black model is violated. The results indicate that the *K-Vol* scenario that violates the underlying assumption, where the selected volatilities for input in the model corresponds to the market volatilities, outperforms the *ATM-Vol* and *AVG-Vol* scenarios, which both have flat volatility structures. The results from the test prove that the increased level of accurate volatility that the more liquid *ATM* swaption could provide does not outperform the shortcomings of choosing a flat volatility structure that does not take the strike offset of the swaption into consideration. By strictly following the assumption of the Black model, the results indicate that by choosing a volatility that corresponds to a more liquid swaption, the hedging performance of the model is improved. However, the conclusion is that the hedging performance in the Black model could not be improved by different selection criteria without violating the underlying assumption of the model.

The second research question takes the approach of comparing the Black model with a model that has a more accurate volatility structure. For this purpose, a model of CEV diffusion class is employed. The results from the comparison support the research question as the hedging performance is improved when employing the CEV model. The conclusion is that a model with a different diffusion class than the Black model can hedge risks across different strikes and expiry dates more properly, without violating the assumption of constant volatility.

9. Suggestions for further research

For future research of the hedging performance of the Black model, one could compare the hedging performance for several other selection criteria for the volatility. For instance, many supporters of the Black model use the implied volatilities of the individual forward contracts and combine the sum of the squares of these volatilities with cross-terms proportional to the correlation between the forward contracts.[31]

One can also evaluate the Black model by setting up a portfolio that is neutral both in delta-gamma and delta-vega exposure. This should consequently reduce the resulting hedging errors, but the hedged position would be more expensive.

Furthermore, the hedging error in the Black model could be compared to other commonly used models, e.g. the SABR model. In this model, the forward swap rate dynamics is of the same type as the CEV model with a stochastic volatility that follows a driftless geometric Brownian motion, possibly instantaneously correlated with the forward swap rate itself.[32] Other interesting models to investigate are those that include a jump-diffusion process. These models can be divided into two different forms. A continuous-time stochastic process modeled by geometric Brownian motion, with small, continuous

[31] See Blanco, Carlos, Josh Gray and Marc Hazzard, 2003, Alternative Valuation Methods for Swaptions: The Devil is in the Details.
[32] See Mercurio, Fabio and Andrea Pallavicini, 2006, Swaption skews and convexity adjustments, p. 5.

random movements. The second form is a discontinuous process modeled by a Poisson distribution.[33] Hence, this model allows for larger forward swap rates movements.

Also, the hedging study could be carried out with rebalancing of the positions in a manner more connected to the hedging strategies practitioners employ. In practice, one typically rebalances a part of the portfolio when the difference between the fixed and floating rate rises over a certain limit and not rebalance on the basis of predetermined time intervals. If one wishes to evaluate the hedging performance from a practitioners view, it could be of interest to test for different rebalancing barriers and to include the transaction costs for rebalancing the portfolio.

[33] See Blanco, Carlos and David Soronow, 2001, Jump diffusion processes – Energy price processes used for derivatives pricing & risk management, p. 84.

10. References

Academic sources

Akume, Daniel, Bernd Luderer and Gerhard-Wilhelm Weber, 2003, Pricing and hedging of swaptions, *Journal of Computational Technologies*, vol. 8, no. 4, pp. 5-14.

Barton, Geoff, Tim Dun and Erik Schlögl, 2001, Simulated swaption delta-hedging in the lognormal forward Libor model, *International Journal of Theoretical Applied Finance*, vol. 4, no. 4, pp. 677-709.

Beckers, Stan, 1980, The Constant Elasticity of Variance Model and Its Implications for Option Pricing, *Journal of Finance*, vol. 35, no. 3, pp. 661-673.

Black, Fischer and Myron Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy*, vol. 81, no. 3, pp. 637-654.

Black, Fischer, 1976, The pricing of commodity contracts, *Journal of financial Economics*, vol. 3, no. 1-2, pp. 167-179.

Blanco, Carlos and David Soronow, 2001, Jump diffusion processes – Energy price processes used for derivatives pricing & risk management, Commodities Now, September, pp. 83-87.

Blanco, Carlos, Josh Gray and Marc Hazzard, 2003, Alternative Valuation Methods for Swaptions: The Devil is in the Details, The Risk Desk, vol. 3, no. 9.

Cole, J. D. and J. Kevorkian, 1981, Perturbation Methods in Applied Mathematics, *Applied Mathematical Sciences*, vol. 34, Springer-Verlag, Berlin and New York.

Cox, John C. and Stephen A. Ross, 1976, The Valuation of Options for Alternative Stochastic Processes, *Journal of Financial Economics*, vol. 3, no. 1-2, pp. 145-166.

Dupire, Bruno, 1993, Pricing and Hedging with Smiles, Working paper, Paribas Capital Markets.

Dupire, Bruno, 1994, Pricing with a Smile, Risk Magazine, vol. 7, no. 1, pp. 18-20.

Geman, Helyette, Nicole El Karoui and Jean-Charles Rochet, 1995, Changes of Numeraire, Changes of Probability Measure and Option Pricing, *Journal of Applied Probability*, vol. 32, no. 2, pp. 443-458.

Hagan, Patrick S. and Diana E. Woodward, 1998, Equivalent Black Volatilities.

Hull, John C., 2006, Options, Futures and Other Derivatives, sixth edition, Pearson Prentice Hall, New Jersey.

Hunt, P.J. and J.E.Kennedy, 2004, Financial derivatives in theory and practice, revised edition, pp. 227-246.

Jamshidian, Farshid, 1997, Libor and swap market models and measures, *Finance and Stochastics*, vol. 1, no. 4, pp. 293-330.

Mercurio, Fabio and Andrea Pallavicini, 2006, Swaption skews and convexity adjustments, Working paper.

Other sources

Bank for International Settlements, http://www.bis.org/statistics/otcder/dt1920a.pdf.

British Bankers' Association, http://www.bbalibor.com.

Appendix A: Swaption hedging results

In this Appendix, the results from the swaption hedging are presented.

Hedging errors sorted on hedging period, *K-Vol -1%*

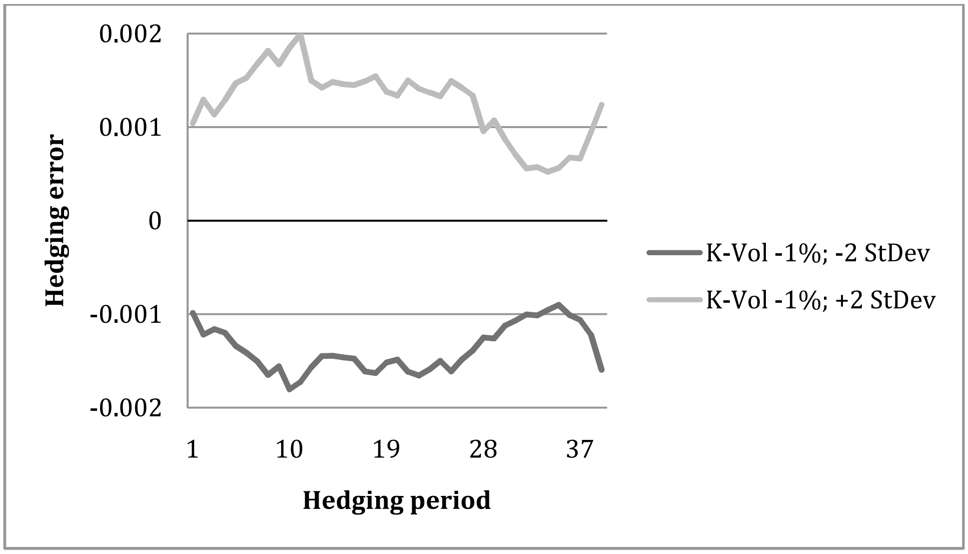

Figure A1: The two lines represent the interval between -2/+2 standard deviations of the hedging errors for each hedging period, where the first period starts three years prior expiry and the last period starts one month prior expiry. The strike rate of the swaptions starts *-1%* of an *ATM* swaption in the first hedging period.

Hedging errors sorted on hedging period, *K-Vol +1%*

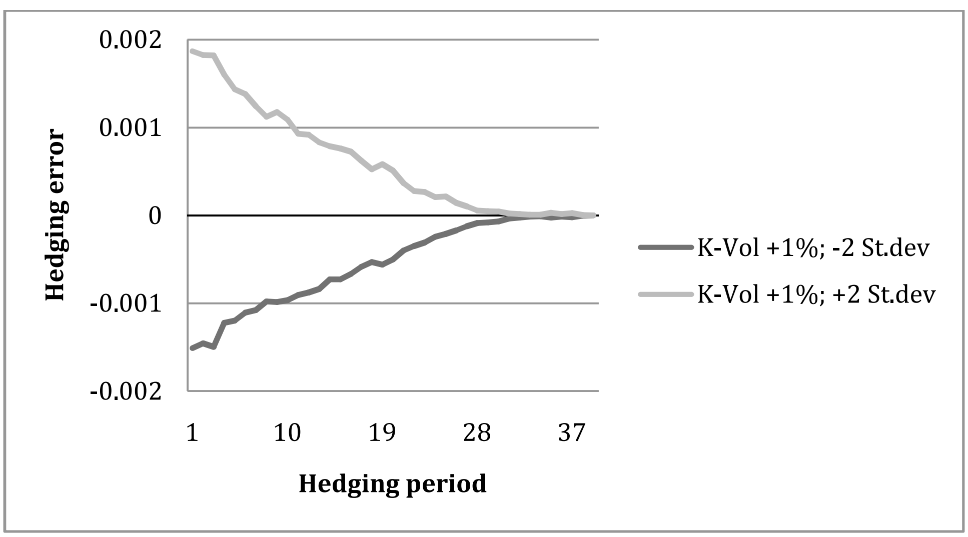

Figure A2: The two lines represent the interval between -2/+2 standard deviations of the hedging errors for each hedging period, where the first period starts three years prior expiry and the last period starts one month prior expiry. The strike rate of the swaptions starts *+1%* of an *ATM* swaption in the first hedging period.

Swaption value for different volatilities w.r.t. the forward swap rate versus the payoff of a payer swaption

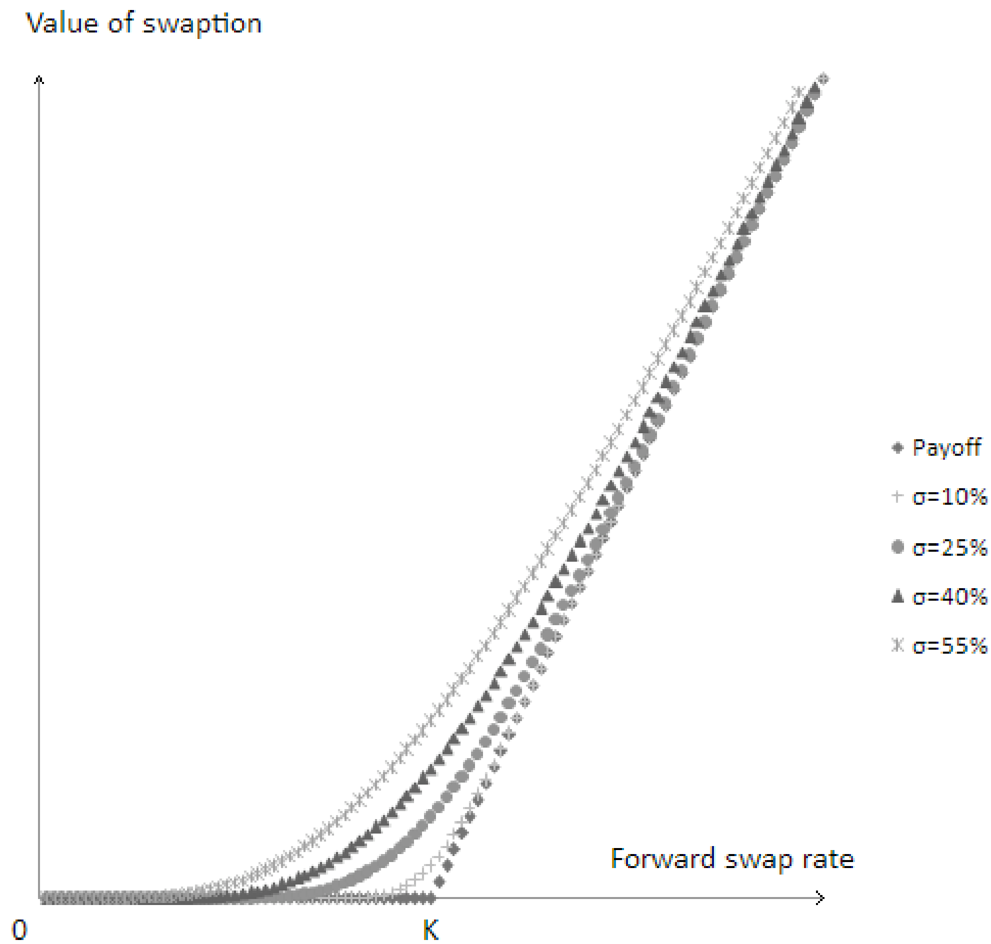

Figure A3: The convex lines plot the swaption value for different volatilities w.r.t. the forward swap rate and the straight line plots the payoff of a payer swaption.

Hedging errors sorted on strike offset
The Black model scenarios

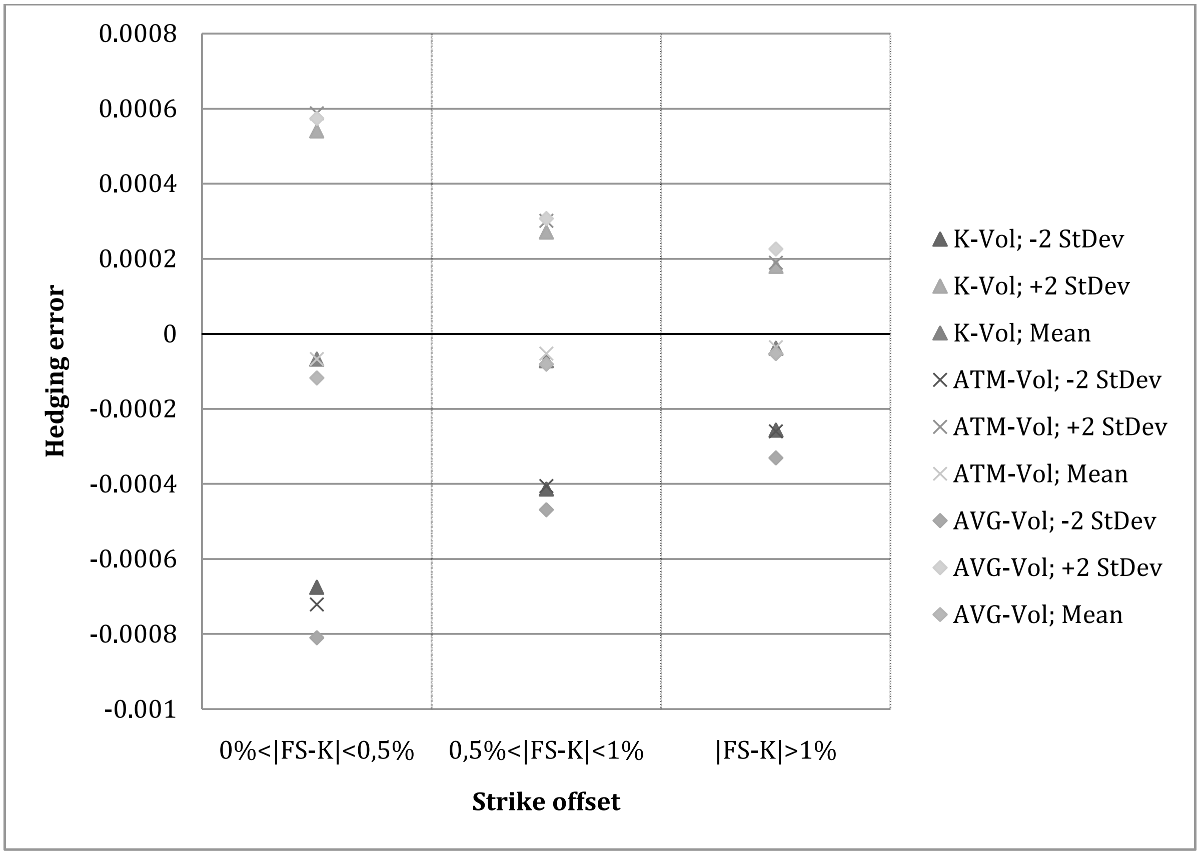

Figure A4: The interval between -2/+2 standard deviations and the mean of the hedging errors for each of the three different volatility scenarios in the Black model are plotted for each strike offset, i.e. the difference between the forward swap rate and the strike rate. The swaptions with strike rates that start *ATM*, *+1%* and *-1%* in the first hedging period are bunched together.

Hedging errors sorted on strike offset
The Black model scenarios

	0%<|FS-K|<0,5%	0,5%<|FS-K|<1%	|FS-K|>1%
***K-Vol*; StDev**	3,04E-04	1,71E-04	1,09E-04
***ATM-Vol*; StDev**	3,27E-04	1,77E-04	1,12E-04
***AVG-Vol*; StDev**	3,46E-04	1,94E-04	1,39E-04

Table A1: The standard deviation of the hedging errors is shown for all three volatility scenarios in the Black model for each strike offset, i.e. the difference between the forward swap rate and the strike rate. The swaptions with strike rates that start *ATM*, *+1%* and *-1%* in the first hedging period are bunched together.

Hedging errors sorted on strike offset
The CEV model scenarios

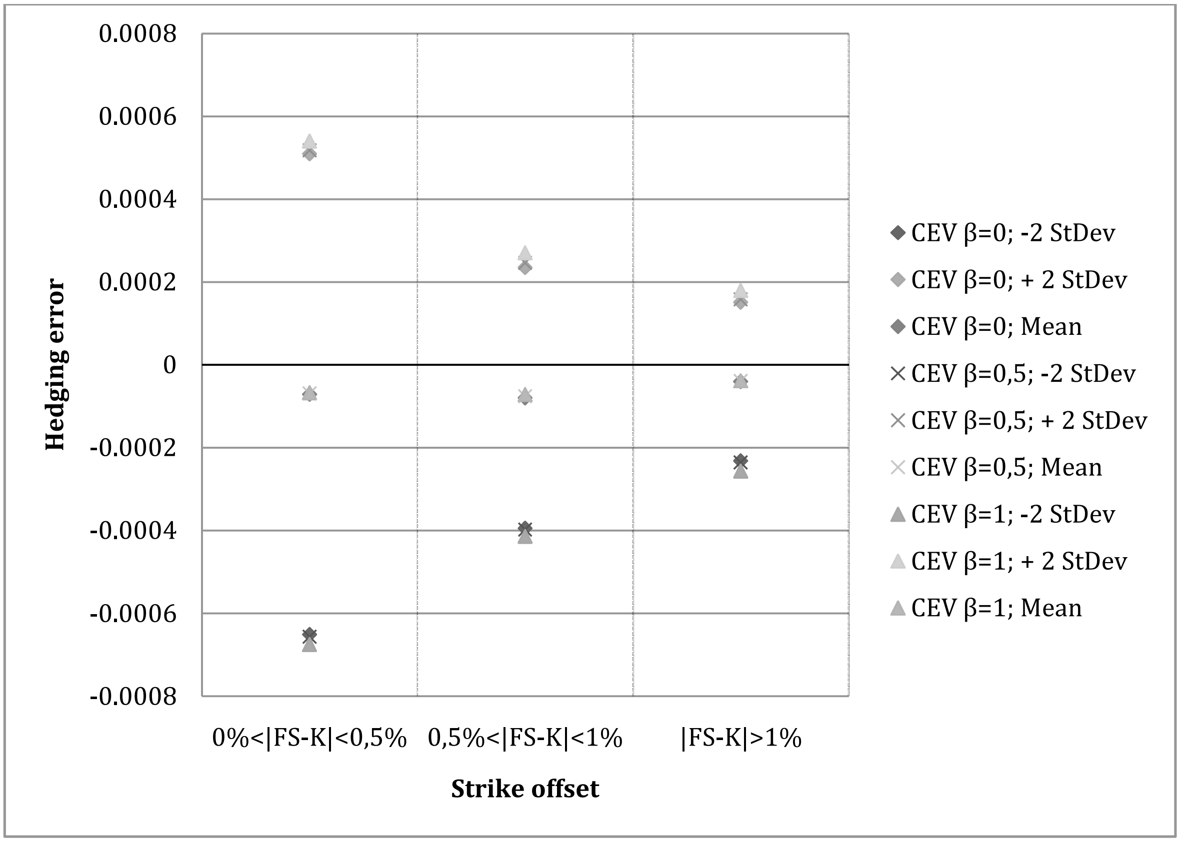

Figure A5: The interval between -2/+2 standard deviations and the mean of the hedging errors for each of the three different β scenarios in the CEV model are plotted for each strike offset, i.e. the difference between the forward swap rate and the strike rate. The swaptions with strike rates that start *ATM*, *+1%* and *-1%* in the first hedging period are bunched together.

Hedging errors sorted on strike offset
The CEV model scenarios

	0%<|FS-K|<0,5%	0,5%<|FS-K|<1%	|FS-K|>1%
CEV β=0; StDev	2,90E-04	1,57E-04	9,55E-05
CEV β=0,5; StDev	2,94E-04	1,61E-04	9,84E-05
CEV β=1; StDev	3,04E-04	1,71E-04	1,09E-04

Table A2: The standard deviation of the hedging errors is shown for all three volatility scenarios in the CEV model for each strike offset, i.e. the difference between the forward swap rate and the strike rate. The swaptions with strike rates that start *ATM*, *+1%* and *-1%* in the first hedging period are bunched together.

Appendix B: Algorithm for estimation in the CEV model

In this Appendix, the algorithm used when estimating the a parameter in the CEV model is presented. Assume the equivalent Black volatility in the CEV model is given by $\sigma_{CEV} = f(FS_t, K, T, a, \beta)$. Then proceed as follows

1. Let $a_1 = \sigma_{Black} \cdot {FS_{t_0}}^{1-\beta}$
2. Let $\sigma_{CEV_1} = f\left(FS_{t_0}, K, T, a_1, \beta\right)$
3. Calculate the ratio $k_1 = \frac{\sigma_{Black}}{\sigma_{CEV_1}}$
4. Let $\sigma_{CEV_2} = f\left(FS_{t_0}, K, T, k_1 \cdot a_1, \beta\right)$

Hopefully, σ_{CEV_2} is now very close to σ_{Black} . If not, continue one or several more times, i.e.

5. Let $a_2 = k_1 \cdot a_1$
6. Calculate the ratio $k_2 = \frac{\sigma_{Black}}{\sigma_{CEV_2}}$
7. Let $\sigma_{CEV_3} = f\left(FS_{t_0}, K, T, k_2 \cdot a_2, \beta\right)$

etc.